

NC $SO(2,3)_*$ gravity: noncommutativity as a source of curvature and torsion

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Abstract

Noncommutative (NC) gravity is constructed on the canonical noncommutative (Moyal-Weyl) space-time as a noncommutative $SO(2,3)_*$ gauge theory. The NC gravity action consists of three different terms: the first term is of Mac-Dowell Mansouri type, while the other two are generalizations of the Einstein-Hilbert action and the cosmological constant term. The expanded NC gravity action is then calculated using the Seiberg-Witten (SW) map and the expansion is done up second order in the deformation parameter. We analyze in details the low energy sector of the full model. We calculate the equations of motion, discuss their general properties and present one solution: the NC correction to Minkowski space-time. Using this solution, we explain breaking of the diffeomorphism symmetry as a consequence of working in a particular coordinate system given by the Fermi normal coordinates.

Keywords: gauge theory of gravity, Seiberg-Witten map, expansion in powers of curvature, NC gravity solutions, Fermi coordinates

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1 Introduction

General Relativity (GR) and Quantum Field Theory (QFT) are leading theories in modern physics. General Relativity successfully describes gravity phenomena from millimeter scale to cosmic scale [1]. On the other hand, Quantum Field Theory remarkably well describes physics at scales from atomic to elementary particle scale [2]. However, a theory that unites these two theories and provides a description of gravity at quantum scales is still missing. One attempt to construct such a theory is the approach of noncommutative (NC) geometry and noncommutative space-time.

During the last twenty years there has been an ongoing effort to try to construct consistent NC gravity models. These models rely on the notion of NC space-time and/or noncommutative geometry and in a certain limit they reduce to General Relativity. One of the main problems in this approach is breaking of diffeomorphism symmetry of General Relativity. Namely, in most of NC gravity models the diffeomorphism symmetry, or at least a part of it, is broken and one needs to understand this breaking and the remaining symmetries (if any). In the following we mention some models of NC gravity. NC gravity via the twist approach [3] is based on the twisted diffeomorphism symmetry. One can write NC Einstein-Hilbert action, derive equations of motion and analyze some particular solutions based on the Killing or semi-Killing twist [4]. However, the full meaning of the twisted symmetry remains to be understood better [5]. In emergent NC gravity models dynamical quantum geometry arises from NC gauge theory given by Yang-Mills matrix models [6]. There are also fuzzy space gravity models and DFR models [7]. Finally, NC gravity can be formulated as a NC gauge theory of Lorentz or (A)dS group using the enveloping algebra approach and the Seiberg-Witten (SW) map [8, 9]. In this approach fermions are easily coupled to gravity and it is straightforward to formulate NC supergravity models [10]. Recently, the SW map approach was related to NC gravity models via the Fedosov deformation quantization of endomorphism bundles [11]. There are also attempts to relate NC gravity models with some testable GR results like gravitational waves, cosmological solutions, Newtonian potential [12].

In this article we construct a NC gravity model following the NC gauge theory approach. We work with the canonical (Moyal-Weyl, θ -constant) noncommutative space-time. However, the model can be straightforwardly generalized to an arbitrary NC space-time coming from an Abelian twist. The main disadvantage of the canonical NC space-time is that, by introducing a constant NC parameter we explicitly break the diffeomorphism symmetry. Therefore, it is natural to ask if this symmetry breaking has some physical explanation. In Section 5 we will provide an explanation of this diffeomorphism breaking. The gauge group of our model is chosen to be the NC $SO(2,3)_*$ group. Motivated by different $f(R)$, $f(T)$ and other modified gravity models we study the SW map expansion of our model and obtain correction terms that are of the first, second, third and fourth order in powers of curvature and torsion. Those terms can be compared with the existing terms in modified gravity models. An advantage of our model is that the relations between different correction terms are not arbitrary but are fixed by the SW map expansion. Calculating NC gravity equations of motion, we

show that noncommutativity is a source of the curvature and torsion. That is, given a flat/torsion-free space-time, noncommutativity induces nonzero curvature/torsion on this space-time. This result is not completely new, it was also discussed in [13] in a different approach to NC gravity. Especially, starting from Minkowski space-time as a solution of commutative vacuum Einstein equations, the corrections induced by our NC gravity model lead to space-time with a constant scalar curvature. Note that this article is a longer and detailed version of [14].

The structure of the article is as follows: In the following section we introduce the full commutative action. For completeness, we repeat the basic notations from our previous papers [15], [16]. After that, the full model consisting of a sum of three different actions is presented. The actions are a MacDovell-Mansouri type of action, a generalization of the Einstein-Hilbert action and the cosmological constant action [17]. The NC generalization of this model is done in Section 3. Using the SW map the second order expansion (in the deformation parameter) of the NC gravity action is calculated. The calculations are long and tedious, so we do not go into details. Instead, we give some of the details in Appendix B. In the zeroth order the NC action reduces to the commutative action containing the Gauss-Bonnet term, Einstein-Hilbert term and the cosmological constant term. The first order correction vanishes, as expected. The first non-vanishing correction is the second order correction. It is given by the terms that are higher order in the curvature and torsion. Since the full second order correction is very complicated, in this paper we only discuss the low energy limit. Therefore, in Section 4 we write the expanded action keeping terms that are of zeroth, first and second order in the derivatives of vierbeins. The equations of motion are then obtained by varying the action with respect to the vierbeins and the spin-connection. NC corrections (θ -dependent terms) appear on the right-hand side of these equations and can be interpreted as sources of curvature and/or torsion. Using these equations of motion, in Section 5 we calculate the NC correction to Minkowski space-time. We see that due to the noncommutativity, Minkowski space-time becomes curved with a constant scalar curvature and the full metric is very close in form to the metric of the AdS space-time. The coordinates in which the solution is given turn out to be Fermi-normal coordinates. This result, its relation with the diffeomorphism symmetry breaking and the work in perspective we discuss in the Conclusions.

2 Commutative model

In this section we review the commutative model. We first repeat the basic notation and then define and discuss the commutative action.

Let us consider a gauge theory on four dimensional Minkowski space-time with the $SO(2, 3)$ group as the gauge group. Note that through the paper we use the "mostly minus" convention for the metric, $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$. See Appendix A for more details on the conventions we use. The gauge field is valued in the $SO(2, 3)$ algebra

$$\omega_\mu = \frac{1}{2}\omega_\mu^{AB}M_{AB}, \quad (2.1)$$

where M_{AB} are the generators of the $SO(2, 3)$ group. The generators satisfy

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}). \quad (2.2)$$

The 5D metric is $\eta_{AB} = \text{diag}(+, -, -, -, +)$. The gauge group indices A, B, \dots take values $0, 1, 2, 3, 5$, while indices a, b, \dots take values $0, 1, 2, 3$. The space-time indices we label with Greek letters. The generators M_{AB} can be defined using the Clifford algebra in 5D. A representation of 5D gamma matrices is obtained from 4D gamma matrices, i. e. $\Gamma_A = (i\gamma_a\gamma_5, \gamma_5)$, where γ_a are 4D gamma matrices. The generators M_{AB} are

$$M_{AB} = \frac{i}{4}[\Gamma_A, \Gamma_B]. \quad (2.3)$$

In the representation given above we obtain

$$\begin{aligned} M_{ab} &= \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab}, \\ M_{5a} &= \frac{1}{2}\gamma_a. \end{aligned} \quad (2.4)$$

Using this representation, the gauge field ω_μ^{AB} can be decomposed as:

$$\omega_\mu = \frac{1}{2}\omega_\mu^{AB}M_{AB} = \frac{1}{4}\omega_\mu^{ab}\sigma_{ab} - \frac{1}{2l}e_\mu^a\gamma_a. \quad (2.5)$$

The parameter l has dimension of length, while fields e_μ^a are dimensionless and ω_μ^{ab} has dimension $1/l$. Under the $SO(2, 3)$ gauge transformations the gauge field transforms as

$$\delta_\epsilon \omega_\mu = \partial_\mu \epsilon + i[\epsilon, \omega_\mu], \quad (2.6)$$

with the gauge parameter denoted by $\epsilon = \frac{1}{2}\epsilon^{AB}M_{AB}$.

The field strength tensor is defined in the standard way as

$$F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu] = \frac{1}{2}F_{\mu\nu}^{AB}M_{AB}. \quad (2.7)$$

Its transformation law under the infinitesimal gauge transformations is given by

$$\delta_\epsilon F_{\mu\nu} = i[\epsilon, F_{\mu\nu}]. \quad (2.8)$$

Just like the gauge potential, the components of the field strength tensor $F_{\mu\nu}^{AB}$ decompose into $F_{\mu\nu}^{ab}$ and $F_{\mu\nu}^{a5}$:

$$F_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu}^{ab} - \frac{1}{l^2}(e_\mu^a e_\nu^b - e_\mu^b e_\nu^a)\right)M_{ab} + F_{\mu\nu}^{a5}M_{a5}, \quad (2.9)$$

where

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac}\omega_\nu^{cb} - \omega_\mu^{bc}\omega_\nu^{ca}, \quad (2.10)$$

$$lF_{\mu\nu}^{a5} = \nabla_\mu e_\nu^a - \nabla_\nu e_\mu^a = T_{\mu\nu}^a. \quad (2.11)$$

The $SO(2,3)$ gauge theory was used in [18] to formulate a gravity theory using the symmetry breaking from $SO(2,3)$ to $SO(1,3)$. Then, using the equations of motion of the model one can identify the fields ω_μ^{ab} with the spin connection and the fields e_μ^a with vierbeins. The fields strengths $R_{\mu\nu}^{ab}$ and $F_{\mu\nu}^{a5} = T_{\mu\nu}^a$ are the curvature tensor and the torsion.

The symmetry breaking was introduced via the scalar field $\phi = \phi^A \Gamma_A$ which transforms in the adjoint representation of the $SO(2,3)$ group,

$$\delta\phi = i[\epsilon, \phi]. \quad (2.12)$$

Using the scalar field ϕ one can write the following gauge invariant actions [17]:

$$S_1 = \frac{il}{64\pi G_N} \text{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \phi, \quad (2.13)$$

$$S_2 = \frac{1}{128\pi G_N l} \text{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} D_\rho \phi D_\sigma \phi \phi + c.c., \quad (2.14)$$

$$S_3 = -\frac{i}{128\pi G_N l} \text{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} D_\mu \phi D_\nu \phi D_\rho \phi D_\sigma \phi \phi, \quad (2.15)$$

with $D_\mu \phi = \partial_\mu \phi - i[\omega_\mu, \phi]$.

We define our commutative model to be the sum of these three actions

$$S = c_1 S_1 + c_2 S_2 + c_3 S_3, \quad (2.16)$$

where c_1, c_2 and c_3 are arbitrary constants that will be determined from some additional constraints. The action (2.16) is invariant under the $SO(2,3)$ gauge symmetry. This symmetry is broken to the $SO(1,3)$ gauge symmetry by choosing $\phi^a = 0, \phi^5 = l$. This choice is sometimes referred to as a physical gauge. After the symmetry breaking the action S_1 reduces to the sum of the Einstein-Hilbert term, the cosmological constant term and the Gauss-Bonnet term:

$$S_1 = -\frac{1}{16\pi G_N} \int d^4x \left(\frac{l^2}{16} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}^{ab} R_{\rho\sigma}^{cd} + eR - \frac{6}{l^2} e \right). \quad (2.17)$$

The action S_2 reduces to the sum of the Einstein-Hilbert term and the cosmological constant term

$$S_2 = -\frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(R - \frac{12}{l^2} \right). \quad (2.18)$$

Finally, the action S_3 reduces to the cosmological constant term only

$$S_3 = -\frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(-\frac{12}{l^2} \right). \quad (2.19)$$

Therefore, after the symmetry breaking our classical action is a sum of these three terms

$$\begin{aligned} S &= c_1 S_1 + c_2 S_2 + c_3 S_3 \\ &= -\frac{1}{16\pi G_N} \int d^4x \left(c_1 \frac{l^2}{16} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}^{ab} R_{\rho\sigma}^{cd} \right. \\ &\quad \left. + \sqrt{-g} ((c_1 + c_2)R - \frac{6}{l^2}(c_1 + 2c_2 + 2c_3)) \right). \end{aligned} \quad (2.20)$$

Now we can partially fix the constants c_1 , c_2 and c_3 by the requirement that the full action after the symmetry breaking reduces to the Einstein Hilbert action with the cosmological constant. The Gauss-Bonnet term is topological, it does not influence the equations of motion and we will not write it further. We choose $c_1 + c_2 = 1$, and the cosmological constant is given by

$$\Lambda = -3 \frac{1 + c_2 + 2c_3}{l^2} .$$

Note that the cosmological constant Λ can be positive, negative or zero, regardless of the symmetry of our model.

3 NC $SO(2, 3)_*$ gravity action

As we have mentioned in Introduction, the NC generalization of General Relativity cannot be formulated in a straightforward way. One of possible ways to achieve this is to use knowledge of the NC gauge theories and treat gravity as a gauge theory of the Poincaré (or AdS or dS) group. In the previous section we defined a rather general model of commutative gravity as a theory with broken $SO(2, 3)$ symmetry (2.20). This model we now generalize to the noncommutative setting.

As in the previous papers [15, 16], we work in the canonical (Moyal-Weyl, θ -constant) NC space-time. Following the approach of deformation quantization we represent non-commutative functions as functions of commuting coordinates and algebra multiplication with the Moyal-Weyl \star -product:

$$f(x) \star g(x) = e^{\frac{i}{2} \theta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\beta}} f(x) g(y) |_{y \rightarrow x} . \quad (3.1)$$

Here $\theta^{\alpha\beta}$ is a constant antisymmetric matrix and its entries are considered to be small deformation parameters¹. The noncommutativity (deformation) is then encoded in the \star -product, while all variables (fields) are functions of commuting coordinates. Integration is well defined since the usual integral is cyclic:

$$\int d^4x (f \star g \star h) = \int d^4x (h \star f \star g) + \text{boundary terms} . \quad (3.2)$$

Assuming that all fields are well behaved at the boundary, these terms vanish and since we are interested in the equations of motion, we will simply ignore the boundary terms throughout this paper. They become important when one discuss conserved quantities or thermodynamics of black holes. The question of boundary terms in the

¹To be more precise, the Moyal-Weyl \star -product should be written as

$$f(x) \star g(x) = e^{\frac{i}{2} \hbar \theta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\beta}} f(x) g(y) |_{y \rightarrow x} ,$$

with the small deformation parameter \hbar and arbitrary constant antisymmetric matrix elements $\theta^{\alpha\beta}$. In the usual notation \hbar is absorbed in the matrix elements $\theta^{\alpha\beta}$ and these are called deformation parameters.

NC gravity action we discussed in details in [16]. In particular, from (3.2) we have $\int d^4x(f \star g) = \int d^4x(g \star f) = \int d^4xfg$. Note that the volume element d^4x is not \star -multiplied with functions under the integral.

In order to construct the NC $SO(2,3)_\star$ gauge theory we use the enveloping algebra approach and the Seiberg-Witten map developed in [19]. We will not go into details of this construction, they can be found in [16]. Here we just write the SW map solutions for the NC gauge field, NC field strength tensor and the NC scalar field in the adjoint representation since we will use them through the paper.

The noncommutative gauge field $\hat{\omega}_\mu$ is defined by the following recursive relation:

$$\hat{\omega}_\mu^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\left(\{\hat{\omega}_\kappa \star \partial_\lambda \hat{\omega}_\mu + \hat{F}_{\lambda\mu}\}\right)^{(n)}, \quad (3.3)$$

where $\hat{\omega}_\mu^{(0)} = \omega_\mu$ is the commutative gauge field and an expression of the type $(A \star B)^{(n)} = A^{(n)}B^{(0)} + A^{(n-1)}B^{(1)} + \dots + A^{(0)}\star^{(1)}B^{(n-1)} + A^{(1)}\star^{(1)}B^{(n-2)} + \dots$ includes all possible terms of order n . Expanding this relation up to first order in the deformation parameter, we find that the NC gauge field $\hat{\omega}_\mu$ is of the form

$$\hat{\omega}_\mu = \omega_\mu - \frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, (\partial_\lambda \omega_\mu + F_{\lambda\mu})\} + \mathcal{O}(\theta^2) \quad (3.4)$$

$$= \frac{1}{4}\omega_\mu^{ab}\sigma_{ab} + \omega_\mu^a\gamma_a + \tilde{\omega}_\mu^a\gamma_a\gamma_5 + \tilde{\omega}_\mu^5\gamma_5 + \omega_\mu I. \quad (3.5)$$

It is obvious from (3.5) that the NC gauge field is valued in the enveloping algebra of the $SO(2,3)$ algebra. However, note that the enveloping algebra in this particular case is finite dimensional. This is one of the advantages of choosing the NC gauge group to be $SO(2,3)_\star$.

The NC field strength tensor is defined as

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu - i[\hat{\omega}_\mu \star \hat{\omega}_\nu] \quad (3.6)$$

and its transformation law under the infinitesimal NC gauge transformations is given by:

$$\delta_\epsilon^\star \hat{F}_{\mu\nu} = i[\hat{\Lambda}_\epsilon \star \hat{F}_{\mu\nu}]. \quad (3.7)$$

Here the NC gauge parameter $\hat{\Lambda}_\epsilon$ is introduced. It is also valued in the enveloping algebra, in zeroth order in the deformation parameter it reduces to the commutative $SO(2,3)$ gauge parameter ϵ and its higher orders can be calculated using the SW map. The SW map solution for $\hat{F}_{\mu\nu}$ follows from the definition (3.6), using the result (3.3). The recursive formula is

$$\begin{aligned} \hat{F}_{\mu\nu}^{(n+1)} &= -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\left(\{\hat{\omega}_\kappa \star \partial_\lambda \hat{F}_{\mu\nu} + D_\lambda \hat{F}_{\mu\nu}\}\right)^{(n)} \\ &\quad + \frac{1}{2(n+1)}\theta^{\kappa\lambda}\left(\{\hat{F}_{\mu\kappa} \star \hat{F}_{\nu\lambda}\}\right)^{(n)}. \end{aligned} \quad (3.8)$$

Note that we do not put a "hat" on the covariant derivative D_μ , the meaning of D_μ is defined by the expression it acts on: $D_\lambda \hat{F}_{\mu\nu} = \partial_\lambda \hat{F}_{\mu\nu} - i[\hat{\omega}_\lambda \star \hat{F}_{\mu\nu}]$ and $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} - i[\omega_\lambda, F_{\mu\nu}]$. One can check that

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda F_{\mu\nu} + D_\lambda F_{\mu\nu}\} + \frac{1}{2}\theta^{\kappa\lambda}\{F_{\mu\kappa}, F_{\nu\lambda}\} + \mathcal{O}(\theta^2) \quad (3.9)$$

$$= \frac{1}{4}F_{\mu\nu}^{ab}\sigma_{ab} + F^a\gamma_a + \tilde{F}^a\gamma_a\gamma_5 + \tilde{F}_{\mu\nu}^5\gamma_5 + F_{\mu\nu}I. \quad (3.10)$$

Finally, the field $\hat{\phi}$ transforms in the adjoint representation

$$\delta_\epsilon^\star \hat{\phi} = i[\hat{\Lambda}_\epsilon \star \hat{\phi}]. \quad (3.11)$$

Using the previous results we find the recursive relation

$$\hat{\phi}^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\left(\{\hat{\omega}_\kappa \star \partial_\lambda \hat{\phi} + D_\lambda \hat{\phi}\}\right)^{(n)}, \quad (3.12)$$

with $D_\lambda \hat{\phi} = \partial_\lambda \hat{\phi} - i[\hat{\omega}_\lambda \star \hat{\phi}]$ and $D_\lambda \phi = \partial_\lambda \phi - i[\omega_\lambda, \phi]$. The solution for $\hat{\phi}$ has the following structure

$$\hat{\phi} = \phi - \frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda \phi + D_\lambda \phi\} + \mathcal{O}(\theta^2) \quad (3.13)$$

$$= \phi^a\gamma_a\gamma_5 + \phi\gamma_5 + \frac{1}{4}\phi^{ab}\sigma_{ab} + \tilde{\phi}^a\gamma_a. \quad (3.14)$$

Having these results at hand, we are now ready to define a NC generalization of the action (2.20). We do it term by term.

3.1 NC generalization of S_1

The NC generalization of the action S_1 (2.13) is given by

$$S_{1NC} = \frac{il}{64\pi G_N} \text{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} \hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma} \star \hat{\phi}. \quad (3.15)$$

The \star -product is the Moyal-Weyl \star -product (3.1), fields with a "hat" are NC fields and we will use the SW map solutions (3.9), (3.13). Using the transformation laws (3.7), (3.11) and the cyclicity of the integral (3.2) one can show that this action is invariant under the NC $SO(2,3)_\star$ gauge transformations. In the limit $\theta^{\alpha\beta} \rightarrow 0$ the action (3.15) reduces to the commutative action (2.13).

The expansion of this action up to the second order in the deformation parameter is done in [16]. The first order correction vanishes. This is an expected result: it was shown in [8] that, if the NC gravity action is real, then the first order (in the deformation parameter) correction has to vanish. This result holds for a wide class of

NC deformations, namely the deformations obtained by an Abelian twist, see [9]. The second order correction is given by:

$$\begin{aligned}
S_{1NC}^{(2)} = & \frac{il}{64\pi G_N} \frac{1}{8} \theta^{\alpha\beta} \theta^{\kappa\lambda} \text{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} \left(\frac{1}{8} \{F_{\alpha\beta}, \{F_{\mu\nu}, F_{\rho\sigma}\}\} \{\phi, F_{\kappa\lambda}\} \right. \\
& - \frac{1}{2} \{F_{\alpha\beta}, \{F_{\rho\sigma}, \{F_{\kappa\mu}, F_{\lambda\nu}\}\}\} \phi - \frac{1}{4} \{\{F_{\mu\nu}, F_{\rho\sigma}\}, \{F_{\kappa\alpha}, F_{\lambda\beta}\}\} \phi \\
& - \frac{i}{4} \{F_{\alpha\beta}, [D_\kappa F_{\mu\nu}, D_\lambda F_{\rho\sigma}]\} \phi - \frac{i}{2} [\{D_\kappa F_{\mu\nu}, F_{\rho\sigma}\}, D_\lambda F_{\alpha\beta}] \phi \\
& - \frac{1}{2} \{F_{\rho\sigma}, \{F_{\alpha\mu}, F_{\beta\nu}\}\} \{\phi, F_{\kappa\lambda}\} + \{\{F_{\alpha\mu}, F_{\beta\nu}\}, \{F_{\kappa\rho}, F_{\lambda\sigma}\}\} \phi \\
& + 2\{F_{\rho\sigma}, \{F_{\beta\nu}, \{F_{\kappa\alpha}, F_{\lambda\mu}\}\}\} \phi + i\{F_{\rho\sigma}, [D_\kappa F_{\alpha\mu}, D_\lambda F_{\beta\nu}]\} \phi \\
& + 2i[\{F_{\beta\nu}, D_\kappa F_{\alpha\mu}\}, D_\lambda F_{\rho\sigma}] \phi \\
& - \frac{i}{4} \{\phi, F_{\kappa\lambda}\} [D_\alpha F_{\mu\nu}, D_\beta F_{\rho\sigma}] - \frac{1}{2} \{D_\kappa D_\alpha F_{\mu\nu}, D_\lambda D_\beta F_{\rho\sigma}\} \phi \\
& + i[\{F_{\kappa\alpha}, D_\lambda F_{\mu\nu}\}, D_\beta F_{\rho\sigma}] \phi + i[\{F_{\lambda\nu}, D_\alpha F_{\kappa\mu}\}, D_\beta F_{\rho\sigma}] \phi \\
& \left. + i[\{F_{\kappa\mu}, D_\alpha F_{\lambda\nu}\}, D_\beta F_{\rho\sigma}] \phi \right). \tag{3.16}
\end{aligned}$$

3.2 NC generalization of S_2

The NC generalization of the action S_2 (2.14) is given by

$$S_{2NC} = \frac{1}{128\pi G_N l} \text{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} \hat{\phi} \star \hat{F}_{\mu\nu} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} + c.c. \tag{3.17}$$

This action is not real so we have to add its complex conjugate by hand. Following the usual steps, we expand (3.17) up to second order in the deformation parameter. The details of the calculation are presented in Appendix B, here we just write the main steps.

Using the formulae (B.1), (B.2) and (B.3) from Appendix B the first order correction follows. It is given by

$$\begin{aligned}
S_{2NC}^{(1)} = & \frac{1}{128\pi G_N l} \text{Tr} \int d^4x \theta^{\alpha\beta} \epsilon^{\mu\nu\rho\sigma} \left(-\frac{1}{4} \phi \{F_{\alpha\beta}, F_{\mu\nu}\} D_\rho \phi D_\sigma \phi \right. \\
& - \frac{i}{2} D_\alpha \phi F_{\mu\nu} (D_\beta D_\rho \phi) D_\sigma \phi - \frac{i}{2} D_\alpha \phi F_{\mu\nu} D_\rho \phi (D_\beta D_\sigma \phi) + \\
& + \frac{1}{2} \phi \{F_{\mu\alpha}, F_{\nu\beta}\} D_\rho \phi D_\sigma \phi + \frac{i}{2} \phi F_{\mu\nu} (D_\alpha D_\rho \phi) (D_\beta D_\sigma \phi) \\
& \left. + \frac{1}{2} \phi F_{\mu\nu} \{F_{\alpha\rho}, D_\beta \phi\} D_\sigma \phi + \frac{1}{2} \phi F_{\mu\nu} D_\rho \phi \{F_{\alpha\sigma}, D_\beta \phi\} \right) + c.c. \tag{3.18}
\end{aligned}$$

Explicit calculation of traces gives $S_2^{(1)} = 0$, so we have to calculate the second order correction. It follows from the first order action as

$$S_{2NC}^{(2)} = \frac{1}{256\pi G_N l} \text{Tr} \int d^4x \theta^{\alpha\beta} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left(-\frac{1}{4} \phi \{ \hat{F}_{\alpha\beta} \star, \hat{F}_{\mu\nu} \} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \phi \right.$$

$$\begin{aligned}
& -\frac{i}{2}\hat{D}_\alpha\phi\star\hat{F}_{\mu\nu}\star(\hat{D}_\beta\hat{D}_\rho\hat{\phi})\star\hat{D}_\sigma\hat{\phi}-\frac{i}{2}\hat{D}_\alpha\hat{\phi}\star\hat{F}_{\mu\nu}\star\hat{D}_\rho\hat{\phi}\star(\hat{D}_\beta\hat{D}_\sigma\hat{\phi}) \quad (3.19) \\
& +\frac{1}{2}\hat{\phi}\star\{\hat{F}_{\mu\alpha}\star\hat{F}_{\nu\beta}\}\star\hat{D}_\rho\hat{\phi}\star\hat{D}_\sigma\hat{\phi}+\frac{i}{2}\hat{\phi}\star\hat{F}_{\mu\nu}\star(\hat{D}_\alpha\hat{D}_\rho\hat{\phi})\star(\hat{D}_\beta\hat{D}_\sigma\hat{\phi}) \\
& +\frac{1}{2}\hat{\phi}\star\hat{F}_{\mu\nu}\star\{\hat{F}_{\alpha\rho}\star\hat{D}_\beta\hat{\phi}\}\star\hat{D}_\sigma\hat{\phi}+\frac{1}{2}\hat{\phi}\star\hat{F}_{\mu\nu}\star\hat{D}_\rho\hat{\phi}\{\hat{F}_{\alpha\sigma}\star\hat{D}_\beta\hat{\phi}\}\Big)^{(1)}+c.c.
\end{aligned}$$

By $()^{(1)}$ it is meant the terms in the bracket are expanded up to first order in the deformation parameter. That includes expansion of the \star -products and the use of the SW map solutions for the corresponding fields.

Using the formulae (B.4-B.7) and the general method outlined in Appendix B, we finally arrive at the second order correction for the NC action S_2

$$\begin{aligned}
S_{2NC}^{(2)} = & \frac{1}{256\pi G_N l} \int d^4x \epsilon^{\mu\nu\rho\sigma} \theta^{\alpha\beta} \theta^{\gamma\delta} \text{Tr} \left(\frac{1}{8} \{F_{\gamma\delta}, \phi\{F_{\alpha\beta}, F_{\mu\nu}\}\} D_\rho\phi D_\sigma\phi \right. \\
& -\frac{i}{4} D_\gamma\phi D_\delta(\{F_{\alpha\beta}, F_{\mu\nu}\}) D_\rho\phi D_\sigma\phi \\
& -\frac{i}{4} \phi [D_\gamma F_{\alpha\beta}, D_\delta F_{\mu\nu}] D_\rho\phi D_\sigma\phi - \frac{1}{4} \phi \{\{F_{\alpha\gamma}, F_{\beta\delta}\}, F_{\mu\nu}\} D_\rho\phi D_\sigma\phi \\
& -\frac{1}{4} \phi \{\{F_{\mu\gamma}, F_{\nu\delta}\}, F_{\alpha\beta}\} D_\rho\phi D_\sigma\phi - \frac{i}{4} \phi \{F_{\alpha\beta}, F_{\mu\nu}\} D_\gamma D_\rho\phi D_\delta D_\sigma\phi \\
& -\frac{1}{4} \phi \{F_{\alpha\beta}, F_{\mu\nu}\} [\{F_{\gamma\rho}, D_\delta\phi\}, D_\sigma\phi] + \frac{i}{4} \{F_{\gamma\delta}, D_\alpha\phi F_{\mu\nu}\} [D_\beta D_\rho\phi, D_\sigma\phi] \\
& +\frac{1}{2} (D_\gamma D_\alpha\phi) D_\delta F_{\mu\nu} [D_\beta D_\rho\phi, D_\sigma\phi] \\
& -\frac{i}{2} \{F_{\gamma\alpha}, D_\delta\phi\} F_{\mu\nu} [D_\beta D_\rho\phi, D_\sigma\phi] - \frac{i}{2} D_\alpha\phi \{F_{\mu\gamma}, F_{\nu\delta}\} [D_\beta D_\rho\phi, D_\sigma\phi] \\
& +\frac{1}{2} D_\alpha\phi F_{\mu\nu} \{D_\gamma D_\beta D_\rho\phi, D_\delta D_\sigma\phi\} - \frac{i}{2} D_\alpha\phi F_{\mu\nu} [\{F_{\gamma\beta}, D_\delta D_\rho\phi\}, D_\sigma\phi] \\
& -\frac{i}{2} D_\alpha\phi F_{\mu\nu} D_\beta([\{F_{\gamma\rho}, D_\delta\phi\}, D_\sigma\phi]) - \frac{1}{4} \{F_{\gamma\delta}, \phi\{F_{\mu\alpha}, F_{\nu\beta}\}\} D_\rho\phi D_\sigma\phi \\
& +\frac{i}{2} D_\gamma\phi D_\delta(\{F_{\mu\alpha}, F_{\nu\beta}\}) D_\rho\phi D_\sigma\phi + \frac{i}{2} \phi [D_\gamma F_{\mu\alpha}, D_\delta F_{\nu\beta}] D_\rho\phi D_\sigma\phi \\
& +\phi \{\{F_{\mu\gamma}, F_{\alpha\delta}\}, F_{\nu\beta}\} D_\rho\phi D_\sigma\phi \\
& +i\phi \{F_{\mu\gamma}, F_{\nu\delta}\} D_\alpha D_\rho\phi D_\beta D_\sigma\phi \\
& +\phi \{F_{\mu\alpha}, F_{\nu\beta}\} [\{F_{\gamma\rho}, D_\delta\phi\}, D_\sigma\phi] - \frac{i}{4} \{F_{\gamma\delta}, \phi F_{\mu\nu}\} D_\alpha D_\rho\phi D_\beta D_\sigma\phi \\
& +\phi F_{\mu\nu} \{F_{\alpha\rho}, D_\beta\phi\} \{F_{\gamma\sigma}, D_\delta\phi\} - \frac{1}{2} D_\gamma\phi D_\delta F_{\mu\nu} D_\alpha D_\rho\phi D_\beta D_\sigma\phi \\
& +i\phi F_{\mu\nu} \{D_\alpha(\{F_{\gamma\rho}, D_\delta\phi\}), D_\beta D_\sigma\phi\} \\
& +\frac{i}{2} \phi F_{\mu\nu} \{\{F_{\gamma\alpha}, D_\delta D_\rho\phi\}, D_\beta D_\sigma\phi\} \\
& -\frac{1}{4} \{F_{\gamma\delta}, \phi F_{\mu\nu}\} [\{F_{\alpha\rho}, D_\beta\phi\}, D_\sigma\phi] \\
& +\frac{i}{2} D_\gamma\phi D_\delta F_{\mu\nu} [\{F_{\alpha\rho}, D_\beta\phi\}, D_\sigma\phi] + \frac{i}{2} \phi F_{\mu\nu} [[D_\gamma F_{\alpha\rho}, D_\delta D_\beta\phi], D_\sigma\phi]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}\phi F_{\mu\nu}D_\gamma(D_\alpha D_\rho\phi)D_\delta(D_\beta D_\sigma\phi) + \frac{1}{2}\phi F_{\mu\nu}[\{\{F_{\alpha\gamma}, F_{\rho\delta}\}, D_\beta\phi\}, D_\sigma\phi] \\
& + \frac{1}{2}\phi F_{\mu\nu}[\{\{F_{\gamma\beta}, D_\delta\phi\}, F_{\alpha\rho}\}, D_\sigma\phi] \Bigg). \tag{3.20}
\end{aligned}$$

3.3 NC generalization of S_3

Finally, we consider the NC generalization of the action S_3 (2.15). Inserting \star -products and promoting the commutative fields to the corresponding NC fields we arrive at:

$$S_{3NC} = -\frac{i}{128\pi G_N l} \text{Tr} \int d^4x \varepsilon^{\mu\nu\rho\sigma} D_\mu \hat{\phi} \star D_\nu \hat{\phi} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} \star \hat{\phi}. \tag{3.21}$$

The zeroth order of the action (3.21) is the commutative action given by (2.15). Following the same steps as in previous subsections and using the formulae from Appendix B we calculate the first order correction to this action:

$$\begin{aligned}
S_{3NC}^{(1)} &= -\frac{i}{128\pi G_N l^3} \theta^{\alpha\beta} \int d^4x \varepsilon^{\mu\nu\rho\sigma} \text{Tr} \Bigg(-\frac{1}{4} \{F_{\alpha\beta}, D_\mu \phi D_\nu \phi\} D_\rho \phi D_\sigma \phi \phi \\
& + \frac{1}{2} \left(\frac{i}{2} (D_\alpha D_\mu \phi) (D_\beta D_\nu \phi) + \frac{1}{2} \{F_{\alpha\mu}, D_\beta \phi\} D_\nu \phi \right. \\
& + \frac{1}{2} D_\mu \phi \{F_{\alpha\nu}, D_\beta \phi\} \Bigg) D_\rho \phi D_\sigma \phi \phi \\
& + D_\mu \phi D_\nu \phi \left(\frac{i}{2} D_\alpha (D_\rho \phi D_\sigma \phi) D_\beta \phi + \frac{i}{2} (D_\alpha D_\rho \phi) (D_\beta D_\sigma \phi) \phi \right. \\
& \left. \left. + \frac{1}{2} \{F_{\alpha\rho}, D_\beta \phi\} D_\sigma \phi \phi + \frac{1}{2} D_\rho \phi \{F_{\alpha\sigma}, D_\beta \phi\} \phi \right) \right). \tag{3.22}
\end{aligned}$$

Again, there is no surprise to find that the calculation of traces leads to $S_{3NC}^{(1)} = 0$. Therefore, the first non-vanishing correction is the second order correction. To calculate it we start from:

$$\begin{aligned}
S_{3NC}^{(2)} &= -\frac{i}{256\pi G_N l^3} \int d^4x \varepsilon^{\mu\nu\rho\sigma} \text{Tr} \Bigg(-\frac{1}{4} \{\hat{F}_{\alpha\beta} \star, \hat{D}_\mu \hat{\phi} \hat{D}_\nu \hat{\phi}\} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} \star \hat{\phi} \\
& + \frac{1}{2} \left(\frac{i}{2} (\hat{D}_\alpha \hat{D}_\mu \hat{\phi}) \star (\hat{D}_\beta \hat{D}_\nu \hat{\phi}) \right. \\
& + \frac{1}{2} \{\hat{F}_{\alpha\mu} \star, \hat{D}_\beta \hat{\phi}\} \star \hat{D}_\nu \hat{\phi} + \frac{1}{2} \hat{D}_\mu \hat{\phi} \star \{\hat{F}_{\alpha\nu} \star, \hat{D}_\beta \hat{\phi}\} \Bigg) \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} \star \hat{\phi} \\
& + \hat{D}_\mu \hat{\phi} \star \hat{D}_\nu \hat{\phi} \star \left(\frac{i}{2} \hat{D}_\alpha (\hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi}) \star \hat{D}_\beta \hat{\phi} + \frac{i}{2} (\hat{D}_\alpha \hat{D}_\rho \hat{\phi}) \star (\hat{D}_\beta \hat{D}_\sigma \hat{\phi}) \star \hat{\phi} \right. \\
& \left. \left. + \frac{1}{2} \{\hat{F}_{\alpha\rho} \star, \hat{D}_\beta \hat{\phi}\} \star \hat{D}_\sigma \hat{\phi} \star \hat{\phi} + \frac{1}{2} \hat{D}_\rho \hat{\phi} \star \{\hat{F}_{\alpha\sigma} \star, \hat{D}_\beta \hat{\phi}\} \star \hat{\phi} \right) \right)^{(1)}. \tag{3.23}
\end{aligned}$$

Explicit calculation then gives:

$$S_{3NC}^{(2)} = -\frac{i}{256\pi G_N l^3} \theta^{\alpha\beta} \theta^{\gamma\delta} \varepsilon^{\mu\nu\rho\sigma} \text{Tr} \int d^4x \left(\frac{1}{32} \{F_{\gamma\delta}, \{F_{\alpha\beta}, D_\mu \phi D_\nu \phi\}\} D_\rho \phi D_\sigma \phi \phi \right.$$

$$\begin{aligned}
& -\frac{1}{8}\left(\frac{i}{2}[D_\gamma F_{\alpha\beta}, D_\delta(D_\mu\phi D_\nu\phi)] + \frac{1}{2}\{\{F_{\alpha\gamma}, F_{\beta\delta}\}, D_\mu\phi D_\nu\phi\}\right. \\
& + \frac{i}{2}\{F_{\alpha\beta}, (D_\gamma D_\mu\phi)(D_\delta D_\nu\phi)\} + \frac{1}{2}\{F_{\alpha\beta}, [D_\mu\phi, \{F_{\gamma\nu}, D_\delta\phi\}]\}) D_\rho\phi D_\sigma\phi \\
& - \frac{1}{8}\{F_{\alpha\beta}, D_\mu\phi D_\nu\phi\}\left(\frac{i}{2}D_\gamma(D_\rho\phi D_\sigma\phi)D_\delta\phi\right. \\
& + \frac{i}{2}(D_\gamma D_\rho\phi)(D_\delta D_\sigma\phi)\phi + \frac{1}{2}[\{F_{\gamma\rho}, D_\delta\phi\}, D_\sigma\phi]\phi) \\
& + \frac{i}{4}\left(-\frac{1}{4}\{F_{\gamma\delta}, (D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)\}D_\rho\phi D_\sigma\phi + \left(\frac{i}{2}(D_\gamma D_\alpha D_\mu\phi)(D_\delta D_\beta D_\nu\phi)\right.\right. \\
& + \frac{1}{2}\{\{F_{\gamma\alpha}, D_\delta D_\mu\phi\}, D_\beta D_\nu\phi\} + \frac{1}{2}\{(D_\alpha\{F_{\gamma\mu}, D_\delta\phi\}), (D_\beta D_\nu\phi)\})D_\rho\phi D_\sigma\phi \\
& + (D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)\left(\frac{i}{2}D_\gamma(D_\rho\phi D_\sigma\phi)D_\delta\phi\right. \\
& + \frac{i}{2}(D_\gamma D_\rho\phi)(D_\delta D_\sigma\phi)\phi + \frac{1}{2}[\{F_{\gamma\rho}, D_\delta\phi\}, D_\sigma\phi]\phi) \\
& + \frac{1}{4}\left(-\frac{1}{4}\{F_{\gamma\delta}, [\{F_{\alpha\mu}, D_\beta\phi\}, D_\nu\phi]\}D_\rho\phi D_\sigma\phi + \left(\frac{i}{2}\{D_\gamma\{F_{\alpha\mu}, D_\beta\phi\}, D_\delta D_\nu\phi\}\right.\right. \\
& + \frac{i}{2}[[D_\gamma F_{\alpha\mu}, D_\delta D_\beta\phi], D_\nu\phi] + \frac{1}{2}[\{\{F_{\alpha\gamma}, F_{\mu\delta}\}, D_\beta\phi\}, D_\nu\phi] \\
& + \frac{1}{2}[\{F_{\alpha\mu}, \{F_{\gamma\beta}, D_\delta\phi\}\}, D_\nu\phi] + \frac{1}{2}[\{F_{\alpha\mu}, D_\beta\phi\}, \{F_{\gamma\nu}, D_\delta\phi\}])D_\rho\phi D_\sigma\phi \\
& + \frac{i}{2}[\{F_{\alpha\mu}, D_\beta\phi\}, D_\nu\phi]\left((D_\gamma(D_\rho\phi D_\sigma\phi)D_\delta\phi + (D_\gamma D_\rho\phi)(D_\delta D_\sigma\phi)\phi\right. \\
& - i[\{F_{\gamma\rho}, D_\delta\phi\}, D_\sigma\phi]\phi) \\
& + \frac{i}{4}\left(-\frac{1}{4}\{F_{\gamma\delta}, D_\mu\phi D_\nu\phi\}[D_\alpha D_\rho\phi, D_\sigma\phi]D_\beta\phi + \left(\frac{i}{2}(D_\gamma D_\mu\phi)(D_\delta D_\nu\phi)\right.\right. \\
& + \frac{1}{2}[\{F_{\gamma\mu}, D_\delta\phi\}, D_\nu\phi])[D_\alpha D_\rho\phi, D_\sigma\phi]D_\beta\phi \\
& + D_\mu\phi D_\nu\phi\left(\frac{i}{2}D_\gamma([D_\alpha D_\rho\phi, D_\sigma\phi])D_\delta D_\beta\phi + \frac{i}{2}\{D_\gamma D_\alpha D_\rho\phi, D_\delta D_\sigma\phi\}D_\beta\phi\right. \\
& + \frac{1}{2}[\{F_{\gamma\alpha}, D_\delta D_\rho\phi\}, D_\sigma\phi]D_\beta\phi + \frac{1}{2}[D_\alpha\{F_{\gamma\rho}, D_\delta\phi\}, D_\sigma\phi]D_\beta\phi \\
& + \frac{1}{2}[D_\alpha D_\rho\phi, \{F_{\gamma\sigma}, D_\delta\phi\}]D_\beta\phi + \frac{1}{2}[D_\alpha D_\rho\phi, D_\sigma\phi]\{F_{\gamma\beta}, D_\delta\phi\}) \\
& + \frac{i}{4}\left(-\frac{1}{4}\{F_{\gamma\delta}, D_\mu\phi D_\nu\phi\}(D_\alpha D_\rho\phi)(D_\beta D_\sigma\phi)\phi + \left(\frac{i}{2}(D_\gamma D_\mu\phi)(D_\delta D_\nu\phi)\right.\right. \\
& + \frac{1}{2}[\{F_{\gamma\mu}, D_\delta\phi\}, D_\nu\phi])(D_\alpha D_\rho\phi)(D_\beta D_\sigma\phi)\phi \\
& + D_\mu\phi D_\nu\phi\left(\frac{i}{2}D_\gamma((D_\alpha D_\rho\phi)(D_\beta D_\sigma\phi))D_\delta\phi + \frac{i}{2}(D_\gamma D_\alpha D_\rho\phi)(D_\delta D_\beta D_\sigma\phi)\phi\right. \\
& + \frac{1}{2}\{\{F_{\gamma\alpha}, D_\delta D_\rho\phi\}, D_\beta D_\sigma\phi\}\phi + \frac{1}{2}\{(D_\alpha\{F_{\gamma\rho}, D_\delta\phi\}), D_\beta D_\sigma\phi\}\phi) \\
& + \frac{i}{4}\left(-\frac{1}{4}\{F_{\gamma\delta}, [\{F_{\alpha\rho}, D_\beta\phi\}, D_\sigma\phi]\}\phi D_\mu\phi D_\nu\phi + \left(\frac{i}{2}\{D_\gamma\{F_{\alpha\rho}, D_\beta\phi\}, D_\delta D_\sigma\phi\}\right.\right. \\
& + \frac{i}{2}[[D_\gamma F_{\alpha\rho}, D_\delta D_\beta\phi], D_\sigma\phi] + \frac{1}{2}[\{\{F_{\alpha\gamma}, F_{\rho\delta}\}, D_\beta\phi\}, D_\sigma\phi] \\
& + \frac{1}{2}[\{F_{\alpha\rho}, \{F_{\gamma\beta}, D_\delta\phi\}\}, D_\sigma\phi] + \frac{1}{2}[\{F_{\alpha\rho}, D_\beta\phi\}, \{F_{\gamma\sigma}, D_\delta\phi\}])\phi D_\mu\phi D_\nu\phi
\end{aligned}$$

$$\begin{aligned}
& +[\{F_{\alpha\rho}, D_\beta\phi\}, D_\sigma\phi]\left(\frac{i}{2}D_\gamma\phi D_\delta(D_\mu\phi D_\nu\phi)\right. \\
& \left. +\frac{i}{2}\phi(D_\gamma D_\mu\phi)(D_\delta D_\nu\phi) + \frac{1}{2}\phi[\{F_{\gamma\mu}, D_\delta\phi\}, D_\nu\phi]\right)\Bigg). \tag{3.24}
\end{aligned}$$

The expanded actions (3.16), (3.20) and (3.24) are obviously invariant under the commutative $SO(2,3)$ gauge transformations, as guaranteed by the SW map.

4 Symmetry breaking and the low energy expansion

The second order expansion of the NC actions (3.15, 3.17, 3.21), given by equations (3.16, 3.20, 3.24) is explicitly invariant under the commutative $SO(2,3)$ gauge symmetry. In order to relate these expanded actions to the General Relativity and its NC corrections, we have to follow the same steps as in the commutative model, Section 2. First we have to break the $SO(2,3)$ gauge symmetry down to the $SO(1,3)$ gauge symmetry (local Lorentz symmetry). Then we have to calculate the traces and write the actions in terms of the geometric quantities (curvature, vierbeins, metric). Let us proceed step by step.

The symmetry breaking is done by choosing the field ϕ to be of the form $\phi = (0, 0, 0, 0, l)$. In this way the zeroth order of the actions (3.15, 3.17, 3.21) reduces to the commutative model with the BG term, Einstein-Hilbert term and the cosmological constant term, (2.16). Then we have to calculate the traces. As we have seen, the first order correction vanishes and the second order correction is the first non-vanishing correction. It is very long and we will not write the full expressions here. Moreover, the expanded actions contain terms that are fourth and lower powers of curvature and second and lower powers of torsion. To analyze the full action is very demanding. Especially, finding equations of motion is a highly non trivial calculation. Additionally, there is no guarantee that the obtained equations of motion will remain second order partial differential equations with respect to the metric and the connection. There are higher order gravity theories, like Lovelock theories, where the equations of motion remain the second order differential equations. In our case, unfortunately it is not clear what will happen with the equations of motion and a careful analysis has to be done.

4.1 Low energy effective NC gravity action

However, we can still analyze different sectors of our model, such as high energy behavior, or low energy behavior, with or without the cosmological constant, etc. In this paper we are interested in the low energy corrections. To be more precise, we keep terms that have at most two derivatives on vierbeins. Therefore, in our analysis we include terms linear in curvature, linear and quadratic in torsion. Additionally, we assume that the spin connection ω_μ^{ab} and first order derivatives of vierbeins such as $\partial_\rho e_\alpha^b$ are of the same order.

The low energy NC correction of the action S_{1NC} is given by

$$\begin{aligned}
S_{1NC}^{(2)} = & -\frac{1}{128\pi G_N l^4} \int d^4x e \theta^{\alpha\beta} \theta^{\gamma\delta} \Big(2R_{\alpha\beta\gamma\delta} - 4R_{\alpha\gamma\beta\delta} + 6g_{\beta\delta} R_{\alpha\mu\gamma}{}^\mu \\
& -\frac{6}{l^2} g_{\alpha\gamma} g_{\beta\delta} - 5T_{\alpha\beta}^a T_{\gamma\delta a} + 10T_{\alpha\gamma}^a T_{\beta\delta a} - 3T_{\alpha\beta\gamma} T_{\delta\mu}{}^\mu - T_{\alpha\beta\rho} T_{\gamma\delta}^\rho - 8T_{\alpha\gamma\delta} T_{\beta\mu}{}^\mu \\
& -2T_{\alpha\gamma\mu} e_\beta^b \nabla_\delta e_\mu^b - 2T_{\alpha\gamma\beta} e_a^\rho \nabla_\delta e_\rho^a + 6T_{\delta\rho\beta} e_a^\rho \nabla_\alpha e_\gamma^a \\
& -2T_{\alpha\beta\delta} e_a^\rho \nabla_\gamma e_\rho^a + T_{\alpha\beta}{}^\mu e_{\delta a} \nabla_\gamma e_\mu^a + 4e_a^\mu e_{b\beta} \nabla_\gamma e_\alpha^a \nabla_\delta e_\mu^b + 4e_a^\mu e_{b\delta} \nabla_\gamma e_\beta^b \nabla_\alpha e_\mu^a \\
& +2g_{\alpha\gamma} e_a^\mu e_b^\nu \nabla_\beta e_\mu^a \nabla_\delta e_\nu^b - 2g_{\alpha\gamma} e_b^\mu e_a^\nu \nabla_\delta e_\nu^b \nabla_\beta e_\mu^a \Big).
\end{aligned}$$

The low energy NC correction of the action S_{2NC} is given by

$$\begin{aligned}
S_{2NC}^{(2)} = & \frac{1}{256\pi G_N l^4} \int d^4x e \theta^{\alpha\beta} \theta^{\gamma\delta} \Big(20R_{\alpha\beta\gamma\delta} - 28R_{\alpha\gamma\beta\delta} - 56g_{\beta\delta} R_{\alpha\mu\gamma}{}^\mu \\
& +\frac{68}{l^2} g_{\alpha\gamma} g_{\beta\delta} + T_{\alpha\beta}^a T_{\gamma\delta a} - 11T_{\alpha\gamma}^a T_{\beta\delta a} + 6T_{\alpha\beta\rho} T_{\rho\gamma\delta} - 16T_{\alpha\gamma\beta} T_{\delta\mu}{}^\mu \\
& +24T_{\alpha\gamma}{}^\rho T_{\rho\delta\beta} + 4g_{\beta\delta} T_{\gamma\sigma}{}^\sigma T_{\alpha\rho}{}^\rho - 4g_{\beta\delta} T_{\gamma\sigma\rho} T_\alpha{}^{\rho\sigma} \\
& +4T_{\alpha\gamma\delta} e_b^\mu \nabla_\beta e_\mu^b + 4T_{\alpha\beta\gamma} e_b^\mu \nabla_\delta e_\mu^b + 2T_{\alpha\beta\rho} e_\gamma^a \nabla_\delta e_\rho^a \\
& -12T_{\alpha\gamma}{}^\rho e_{\delta a} \nabla_\beta e_\rho^a - 28T_{\alpha\mu\gamma} e_a^\mu \nabla_\beta e_\delta^a + 4g_{\beta\gamma} T_{\alpha\rho}{}^\sigma e_b^\rho \nabla_\delta e_\sigma^b - 4g_{\alpha\gamma} T_{\nu\beta}{}^\nu e_\mu^a \nabla_\delta e_\mu^a \\
& -40e_a^\rho e_{\delta b} \nabla_\alpha e_\gamma^a \nabla_\beta e_\rho^b - 12g_{\beta\delta} e_b^\mu e_a^\nu \nabla_\gamma e_\mu^b \nabla_\alpha e_\nu^a \\
& +32e_b^\mu e_{\delta a} \nabla_\alpha e_\gamma^a \nabla_\beta e_\mu^b + 12g_{\beta\delta} e_c^\mu e_a^\nu \nabla_\alpha e_\rho^a \nabla_\gamma e_\mu^c \Big).
\end{aligned}$$

The low energy NC correction of the action S_{3NC} is given by

$$\begin{aligned}
S_{3NC}^{(2)} = & \int d^4x \frac{e \theta^{\alpha\beta} \theta^{\gamma\delta}}{128\pi G_N l^4} \Big(38R_{\alpha\beta\gamma\delta} - 44R_{\alpha\gamma\beta\delta} - 36R_{\alpha\gamma} g_{\beta\delta} + \frac{56}{l^2} g_{\alpha\gamma} g_{\beta\delta} \\
& -7T_{\alpha\beta}^a T_{\gamma\delta a} + 14T_{\alpha\gamma}^a T_{\beta\delta a} - 2T_{\alpha\beta\gamma} T_{\delta\rho}^\rho + 4T_{\alpha\gamma}^\rho T_{\delta\rho\beta} + 4g_{\beta\delta} T_{\alpha\rho}^\rho T_{\gamma\sigma}^\sigma \\
& -4g_{\beta\delta} T_{\alpha\rho}^\sigma T_{\gamma\sigma}^\rho + 32\nabla_\alpha e_\gamma^a e_{\delta a} \nabla_\beta e_\rho^b e_b^\rho - 32\nabla_\alpha e_\gamma^a e_a^\rho \nabla_\beta e_\rho^b e_{\delta b}^b \\
& +8g_{\beta\delta} \nabla_\alpha e_\rho^a e_a^\sigma \nabla_\gamma e_\sigma^b e_b^\rho - 8g_{\beta\delta} \nabla_\alpha e_\rho^a e_a^\rho \nabla_\gamma e_\sigma^b e_b^\sigma - 12T_{\alpha\gamma}^\rho \nabla_\beta e_\rho^a e_{\delta a}^a \\
& +18T_{\alpha\beta\gamma} \nabla_\delta e_\rho^a e_a^\rho - 8T_{\alpha\gamma\beta} \nabla_\delta e_\rho^a e_a^\rho + 16T_{\gamma\rho\beta} \nabla_\alpha e_\delta^a e_a^\rho \\
& -4T_{\alpha\rho}^\sigma \nabla_\gamma e_\sigma^a e_a^\rho g_{\beta\delta} + 4T_{\alpha\rho}^\rho \nabla_\gamma e_\sigma^a e_a^\sigma g_{\beta\delta} \Big). \tag{4.1}
\end{aligned}$$

Remembering that $c_1 + c_2 = 1$ the resulting action follows

$$\begin{aligned}
S_{NC} = & -\frac{1}{16\pi G_N} \int d^4x e \Big(R - \frac{6}{l^2} (1 + c_2 + 2c_3) \Big) \\
& +\frac{1}{128\pi G_N l^4} \int d^4x e \theta^{\alpha\beta} \theta^{\gamma\delta} \Big((-2 + 12c_2 + 38c_3) R_{\alpha\beta\gamma\delta} \\
& + (4 - 18c_2 - 44c_3) R_{\alpha\gamma\beta\delta} - (6 + 22c_2 + 36c_3) g_{\beta\delta} R_{\alpha\mu\gamma}{}^\mu \\
& +\frac{6 + 28c_2 + 56c_3}{l^2} g_{\alpha\gamma} g_{\beta\delta} + (5 - \frac{9}{2}c_2 - 7c_3) T_{\alpha\beta}^a T_{\gamma\delta a} \Big) \tag{4.2}
\end{aligned}$$

$$\begin{aligned}
& +(-10 + \frac{9}{2}c_2 + 14c_3)T_{\alpha\gamma}^a T_{\beta\delta a} + (3 - 3c_2 - 2c_3)T_{\alpha\beta\gamma} T_{\delta\mu}^\mu \\
& + (1 + 2c_2)T_{\alpha\beta\rho} T_{\gamma\delta}^\rho + 8T_{\alpha\gamma\delta} T_{\beta\mu}^\mu \\
& - (2c_2 + 4c_3)T_{\alpha\gamma\rho} T_{\delta\beta}^\rho + (2c_2 + 4c_3)g_{\beta\delta} T_{\gamma\sigma}^\sigma T_{\alpha\rho}^\rho \\
& - (2c_2 + 4c_3)T_{\alpha\rho\sigma} T_{\gamma}^{\sigma\rho} g_{\beta\delta} + (-2 + 4c_2 + 18c_3)T_{\alpha\beta\gamma} e_a^\rho \nabla_\delta e_\rho^a \\
& + (6 - 8c_2 - 8c_3)T_{\alpha\gamma\beta} e_a^\rho \nabla_\delta e_\rho^a + (2 + 4c_2 + 12c_3)T_{\alpha\gamma}^\mu e_\beta^a \nabla_\delta e_\mu^a \\
& - T_{\alpha\beta}^\mu e_\delta^a \nabla_\gamma e_\mu^a + (-6 - 8c_2 - 16c_3)T_{\delta\rho\beta} e_a^\rho \nabla_\alpha e_\gamma^a \\
& - (2c_2 + 4c_3)g_{\alpha\gamma} T_{\mu\beta}^\mu e_a^\rho \nabla_\delta e_\rho^a - (2c_2 + 4c_3)g_{\beta\delta} T_{\alpha\rho}^\sigma e_a^\rho \nabla_\gamma e_\sigma^a \\
& - (4 + 16c_2 + 32c_3)e_a^\mu e_{b\beta} \nabla_\gamma e_\alpha^a \nabla_\delta e_\mu^b + (4 + 12c_2 + 32c_3)e_{\delta a} e_b^\mu \nabla_\alpha e_\gamma^a \nabla_\beta e_\mu^b \\
& - (2 + 4c_2 + 8c_3)g_{\beta\delta} e_a^\mu e_b^\nu \nabla_\gamma e_\mu^a \nabla_\alpha e_\nu^b + (2 + 4c_2 + 8c_3)g_{\beta\delta} e_a^\mu e_c^\rho \nabla_\alpha e_\rho^a \nabla_\gamma e_\mu^c.
\end{aligned}$$

To obtain this action we used that $D_\alpha F_{\mu\nu}$ is the $SO(2,3)$ covariant derivative and its components are

$$\begin{aligned}
(D_\alpha F_{\mu\nu})^{ab} &= \nabla_\alpha F_{\mu\nu}^{ab} - \frac{1}{l^2}(e_\alpha^a T_{\mu\nu}^b - e_\alpha^b T_{\mu\nu}^a), \\
(D_\alpha F_{\mu\nu})^{a5} &= \frac{1}{l}(\nabla_\alpha T_{\mu\nu}^a + e_\alpha^m F_{\mu\nu m}^a), \\
(D_\kappa D_\alpha F_{\mu\nu})^{ab} &= \nabla_\kappa \nabla_\alpha F_{\mu\nu}^{ab} - \frac{1}{l^2}\left((\nabla_\kappa e_\alpha^a)T_{\mu\nu}^b - (\nabla_\kappa e_\alpha^b)T_{\mu\nu}^a + e_\alpha^a(\nabla_\kappa T_{\mu\nu}^b) \right. \\
&\quad \left. - e_\alpha^b(\nabla_\kappa T_{\mu\nu}^a) + e_\kappa^a(\nabla_\alpha T_{\mu\nu}^b) - e_\kappa^b(\nabla_\alpha T_{\mu\nu}^a) + e_\kappa^a e_\alpha^m F_{\mu\nu m}^b - e_\kappa^b e_\alpha^m F_{\mu\nu m}^a\right),
\end{aligned}$$

with the $SO(1,3)$ covariant derivative

$$\begin{aligned}
\nabla_\alpha F_{\mu\nu}^{ab} &= \partial_\alpha F_{\mu\nu}^{ab} + \omega_\alpha^{ac} F_{\mu\nu c}^b - \omega_\alpha^{bc} F_{\mu\nu c}^a, \\
\nabla_\alpha T_{\mu\nu}^a &= \partial_\alpha T_{\mu\nu}^a + \omega_\alpha^{ac} T_{\mu\nu c}^a.
\end{aligned}$$

We also used that

$$\begin{aligned}
(D_\alpha \phi)^a &= e_\alpha^a, \\
(D_\alpha \phi)^5 &= 0, \\
(D_\alpha D_\beta \phi)^a &= (\nabla_\alpha e_\beta)^a, \\
(D_\alpha D_\beta \phi)^5 &= -\frac{1}{l}g_{\alpha\beta}.
\end{aligned}$$

Before we determine the equations of motion, let us briefly discuss the action (4.2). We see that this action is invariant under the $SO(1,3)$ gauge symmetry. However, due to the noncommutativity it is no longer invariant under the diffeomorphism symmetry. The non-invariant terms manifest themselves in two ways. Firstly, there are tensors contracted with the NC parameter $\theta^{\alpha\beta}$ such as $\theta^{\alpha\beta}\theta^{\kappa\lambda}R_{\alpha\kappa\beta\lambda}$. Since $\theta^{\alpha\beta}$ is not a tensor under the diffeomorphism symmetry (it is a constant matrix that does not transform under the diffeomorphism), those terms are also not scalars (tensors). Then there are

terms in which $SO(1,3)$ covariant derivatives of vierbeins appear. Using the metricity condition

$$\nabla_\mu^{tot} e_\rho^a = \partial_\mu e_\rho^a + \omega_\mu^{ab} e_{\rho b} - \Gamma_{\mu\rho}^\sigma e_\sigma^a = 0 \quad (4.3)$$

the $SO(1,3)$ covariant derivative can be written as

$$\nabla_\mu e_\rho^a = \partial_\mu e_\rho^a + \omega_\mu^{ab} e_{\rho b} = \Gamma_{\mu\rho}^\sigma e_\sigma^a. \quad (4.4)$$

Therefore, the affine connection $\Gamma_{\mu\rho}^\sigma$ appears explicitly in (4.2). Note that this affine connection does not have to be given by the Christoffel symbols. We will see in Section 6 that the noncommutativity can generate the antisymmetric part of the connection, leading to the appearance of torsion. Some of the terms with the explicit $\Gamma_{\mu\rho}^\sigma$ s can be grouped to form the curvature tensor, but some will remain and make the diffeomorphism non-invariance explicit.

4.2 Low energy equations of motion

The equations of motions are obtained by varying the action (4.2) with respect to the vierbein and the spin connection. Some useful formulae are given in Appendix C. In this article we are interested in NC corrections to the GR solutions with vanishing torsion. Therefore, in the equations of motion we impose the condition $T_{\mu\nu}^a = 0$. A more general form of the equations of motion will be presented in future work.

Finally, the equation of motion for the vierbein is given by

$$R_{\alpha\gamma}^{cd} e_d^\gamma e_a^\alpha e_c^\mu - \frac{1}{2} e_a^\mu R + \frac{3}{l^2} (1 + c_2 + 2c_3) e_a^\mu = \tau_a^\mu, \quad (4.5)$$

where

$$\begin{aligned} \tau_a^\mu &= -\frac{8\pi G_N}{e} \frac{\delta S_{NC}^{(2)}}{\delta e_\mu^a} \\ &= -\frac{\theta^{\alpha\beta}\theta^{\gamma\delta}}{16l^4} \Big((-4 + 6c_2 + 22c_3) e_a^\mu R_{\alpha\beta\gamma\delta} \\ &\quad + (4 - 18c_2 + 44c_3) (e_a^\mu R_{\alpha\gamma\beta\delta} - 2\delta_\alpha^\mu R_{\beta\delta\gamma a}) \\ &\quad - (6 + 22c_2 + 36c_3) e_a^\mu g_{\beta\delta} R_{\alpha\gamma} + 2\delta_\alpha^\mu e_{\gamma a} R_{\beta\delta} + g_{\beta\delta} R_{\gamma\lambda} e^\lambda_\alpha \delta_\alpha^\mu - g_{\beta\delta} R_{\alpha\lambda\gamma}^\mu e_a^\lambda \\ &\quad + (4 + 16c_2 + 32c_3) (e_c^\mu e_{\gamma b} e_a^\rho \nabla_\beta e_\delta^c \nabla_\alpha e_\rho^b \\ &\quad - e_a^\mu e_{\beta b} e_c^\rho \nabla_\gamma e_\alpha^c \nabla_\delta e_\rho^b - \delta_\alpha^\mu e_b^\rho \nabla_\gamma e_{\rho a} \nabla_\delta e_\beta^b) \\ &\quad + (2 + 2c_2 + 4c_3) g_{\beta\delta} e_a^\mu e_c^\rho e_d^\sigma (\nabla_\alpha e_\rho^c \nabla_\gamma e_\sigma^d - \nabla_\alpha e_\rho^d \nabla_\gamma e_\sigma^c) \\ &\quad + (4 + 6c_2 + 12c_3) (e_a^\mu g_{\beta\delta} e_b^\rho \nabla_\alpha \nabla_\gamma e_\rho^b - e_{\delta b} e_c^\rho e_a^\mu \nabla_\beta e_\rho^c \nabla_\gamma e_\alpha^b \\ &\quad + g_{\beta\delta} e_a^\rho e_b^\sigma e_c^\mu \nabla_\alpha e_\sigma^c \nabla_\gamma e_\rho^b - g_{\beta\delta} e_a^\rho e_b^\mu \nabla_\alpha \nabla_\gamma e_\rho^b) \\ &\quad - (4 + 12c_2 + 32c_3) e_a^\mu \nabla_\beta e_{\delta b} \nabla_\gamma e_\alpha^b + (7 - 14c_2 - 54c_3) \delta_\alpha^\mu R_{\gamma\delta\beta a} \\ &\quad + (5 + 12c_2 + 24c_3) (2e_{\gamma a} e_b^\mu e_c^\sigma \nabla_\delta e_\beta^b \nabla_\alpha e_\sigma^c + 2e_b^\mu \nabla_\alpha e_{\gamma a} \nabla_\delta e_\beta^b \\ &\quad - 2e_{\gamma a} e_b^\sigma e_c^\mu \nabla_\alpha e_\sigma^c \nabla_\delta e_\beta^b - e_{\delta a} R_{\alpha\beta}^\mu{}_\gamma) \end{aligned}$$

$$\begin{aligned}
& + (2 + 8c_2 - 12c_3)\delta_\alpha^\mu e_{\delta a}(e_c^\sigma e_b^\rho \nabla_\beta e_\sigma^c \nabla_\gamma e_\rho^b - e_b^\sigma e_c^\rho \nabla_\beta e_\sigma^c \nabla_\gamma e_\rho^b) \\
& + (6 + 24c_2 + 36c_3)\delta_\alpha^\mu e_b^\rho \nabla_\gamma e_\rho^b \nabla_\beta e_{\delta a} + 2(-2 + 4c_2 + 18c_3)\delta_\alpha^\mu e_{\gamma a} e_b^\rho \nabla_\beta \nabla_\delta e_\rho^b \\
& + (2 + 4c_2 + 12c_3)\delta_\alpha^\mu e_{\gamma b} e_a^\rho \nabla_\delta \nabla_\beta e_\rho^b \\
& - (6 + 8c_2 + 16c_3)\delta_\alpha^\mu (e_{\gamma a} e_c^\sigma e_b^\rho \nabla_\delta e_\beta^b \nabla_\rho e_\sigma^c + e_d^\rho \nabla_\delta e_\beta^d \nabla_\rho e_{\gamma a} \\
& - e_{\gamma a} e_c^\rho e_d^\sigma \nabla_\rho e_\sigma^c \nabla_\delta e_\beta^d + e_{\gamma a} e_d^\rho \nabla_\rho \nabla_\delta e_\beta^d) \\
& + (6 - 8c_2 - 8c_3)\delta_\alpha^\mu e_{\gamma a} e_d^\rho \nabla_\delta \nabla_\beta e_\rho^d \\
& + (2 + 2c_2 + 8c_3)\delta_\alpha^\mu e_{\delta b}(e_c^\rho e_a^\sigma \nabla_\gamma e_\sigma^c \nabla_\rho e_\beta^b - e_c^\sigma e_a^\rho \nabla_\gamma e_\sigma^c \nabla_\rho e_\beta^b) \\
& + (6 + 22c_2 + 48c_3)\delta_\alpha^\mu e_{\beta b}(e_c^\sigma e_a^\rho \nabla_\gamma e_\sigma^c \nabla_\delta e_\rho^b - e_a^\sigma e_c^\rho \nabla_\gamma e_\sigma^c \nabla_\delta e_\rho^b) \\
& - 2\delta_\alpha^\mu (e_c^\sigma e_a^\rho e_{\delta b} \nabla_\beta e_\sigma^c \nabla_\gamma e_\rho^b - e_c^\rho e_a^\sigma e_{\delta b} \nabla_\beta e_\sigma^c \nabla_\gamma e_\rho^b - e_a^\rho e_\gamma^b \nabla_\beta \nabla_\delta e_\rho^b) \\
& - (8 + 20c_2 + 44c_3)\delta_\alpha^\mu e_a^\rho \nabla_\gamma e_\rho^b \nabla_\beta e_\delta^b \\
& + (2c_2 + 4c_3)g_{\beta\delta}(\delta_\alpha^\mu e_b^\rho e_a^\sigma \nabla_\sigma \nabla_\gamma e_\rho^b \\
& - \delta_\alpha^\mu e_c^\sigma e_a^\rho \nabla_\sigma \nabla_\gamma e_\rho^c - \delta_\alpha^\mu e_c^\nu e_a^\rho e_d^\sigma \nabla_\nu e_\delta^c \nabla_\gamma e_\rho^d \\
& - \delta_\alpha^\mu e_d^\nu e_c^\sigma \nabla_\rho e_\nu^d \nabla_\gamma e_\sigma^c - e_a^\rho e_b^\nu e_c^\sigma \nabla_\nu e_\rho^c \nabla_\gamma e_\sigma^b - e_a^\sigma e_c^\nu e_d^\rho \nabla_\rho e_\nu^d \nabla_\gamma e_\sigma^c) \\
& + \frac{6 + 28c_2 + 56c_3}{l^2}(g_{\alpha\gamma}g_{\beta\delta}e_a^\mu + 4g_{\beta\delta}\delta_\alpha^\mu e_\gamma^a) \Big) . \tag{4.6}
\end{aligned}$$

Multiplying the previous equation with e_a^ν and using the metricity condition we obtain

$$R^{\nu\mu} - \frac{1}{2}g^{\mu\nu}R + \frac{3}{l^2}(1 + c_2 + 2c_3)g^{\mu\nu} = \tau^{\mu\nu}, \tag{4.7}$$

with

$$\begin{aligned}
\tau^{\mu\nu} = & -\frac{\theta^{\alpha\beta}\theta^{\gamma\delta}}{16l^4}\Big((-4 + 6c_2 + 22c_3)R_{\alpha\beta\gamma\delta}g^{\mu\nu} - (7 - 14c_2 - 54c_3)R_{\beta\gamma\delta}^\nu\delta_\alpha^\mu \\
& + (4 - 18c_2 - 44c_3)(2R_{\gamma\beta\delta}^\nu\delta_\alpha^\mu + R_{\alpha\gamma\beta\delta}g^{\mu\nu}) \\
& - (5 + 12c_2 + 24c_3)R_{\gamma\alpha\beta}^\mu\delta_\delta^\nu \\
& - (6 + 22c_2 + 36c_3)(g^{\mu\nu}g_{\beta\delta}R_{\alpha\gamma} + 2\delta_\beta^\mu\delta_\delta^\nu R_{\alpha\gamma} + g_{\beta\delta}\delta_\gamma^\mu R_{\alpha}^\nu - g_{\beta\delta}R_{\alpha}^\nu{}_\gamma^\mu) \\
& + (4 + 16c_2 + 32c_3)(g_{\sigma\beta}g^{\rho\nu}\Gamma_{\alpha\gamma}^\mu\Gamma_{\delta\rho}^\sigma - g^{\mu\nu}g_{\rho\beta}\Gamma_{\alpha\gamma}^\sigma\Gamma_{\sigma\delta}^\rho) \\
& + (2 + 2c_2 + 4c_3)g^{\mu\nu}g_{\delta\beta}\Gamma_{\alpha\sigma}^\sigma\Gamma_{\gamma\rho}^\rho - (4 + 12c_2 + 32c_3)g^{\mu\nu}g_{\rho\sigma}\Gamma_{\beta\delta}^\rho\Gamma_{\gamma\alpha}^\sigma \\
& + (2 + 4c_2 + 8c_3)g^{\mu\nu}g_{\beta\delta}\Gamma_{\gamma\rho}^\sigma\Gamma_{\alpha\sigma}^\rho \\
& + (4 + 6c_2 + 12c_3)(g^{\mu\nu}(g_{\beta\delta}\partial_\alpha\Gamma_{\gamma\rho}^\rho - g_{\delta\rho}\Gamma_{\beta\sigma}^\sigma\Gamma_{\alpha\gamma}^\rho) \\
& + g^{\rho\nu}(g_{\alpha\sigma}\Gamma_{\beta\delta}^\sigma\Gamma_{\gamma\rho}^\mu + g_{\alpha\delta}\partial_\beta\Gamma_{\gamma\rho}^\mu)) \\
& + (10 + 24c_2 + 48c_3)(\delta_\gamma^\nu\Gamma_{\alpha\rho}^\rho\Gamma_{\beta\delta}^\mu + \Gamma_{\delta\beta}^\mu\Gamma_{\alpha\gamma}^\nu - \delta_\gamma^\nu\Gamma_{\alpha\sigma}^\mu\Gamma_{\beta\delta}^\sigma) \\
& - (4 + 4c_2 + 8c_3)g_{\beta\delta}g^{\sigma\nu}\Gamma_{\gamma\rho}^\rho\Gamma_{\alpha\sigma}^\mu + (6 + 24c_2 + 36c_3)\delta_\alpha^\mu\Gamma_{\gamma\rho}^\rho\Gamma_{\beta\delta}^\nu \\
& + (2 + 8c_2 - 12c_3)\delta_\alpha^\mu\delta_\delta^\nu(\Gamma_{\beta\sigma}^\sigma\Gamma_{\gamma\rho}^\rho - \Gamma_{\beta\sigma}^\rho\Gamma_{\gamma\rho}^\sigma) \\
& + (-4 + 8c_2 + 36c_3)\delta_\gamma^\nu\delta_\alpha^\mu\partial_\beta\Gamma_{\delta\rho}^\rho + (6 - 8c_2 - 8c_3)\delta_\gamma^\nu\delta_\alpha^\mu\partial_\delta\Gamma_{\beta\delta}^\rho \\
& + (2 + 28c_3)\delta_\alpha^\mu\delta_\gamma^\nu\Gamma_{\delta\rho}^\sigma\Gamma_{\beta\sigma}^\rho + 2\delta_\alpha^\mu g^{\rho\nu}g_{\gamma\kappa}(\partial_\beta\Gamma_{\delta\rho}^\kappa + \Gamma_{\delta\rho}^\sigma\Gamma_{\beta\sigma}^\kappa)
\end{aligned}$$

$$\begin{aligned}
& -(6 + 8c_2 + 16c_3)\delta_\alpha^\mu(\delta_\gamma^\nu\Gamma_{\rho\sigma}^\sigma\Gamma_{\beta\delta}^\rho + \Gamma_{\rho\gamma}^\nu\Gamma_{\beta\delta}^\rho + \delta_\gamma^\nu\partial_\rho\Gamma_{\beta\delta}^\rho) \\
& + (2 + 2c_2 + 8c_3)\delta_\alpha^\mu g^{\rho\nu}g_{\tau\delta}(-\Gamma_{\gamma\sigma}^\sigma\Gamma_{\beta\rho}^\tau + \Gamma_{\gamma\rho}^\sigma\Gamma_{\beta\sigma}^\tau) \\
& + (6 + 22c_2 + 48c_3)\delta_\alpha^\mu g^{\rho\nu}g_{\tau\beta}(\Gamma_{\gamma\sigma}^\sigma\Gamma_{\delta\rho}^\tau - \Gamma_{\gamma\rho}^\sigma\Gamma_{\delta\sigma}^\tau) \\
& - 2\delta_\alpha^\mu g_{\kappa\delta}g^{\rho\nu}(\Gamma_{\beta\tau}^\tau\Gamma_{\gamma\rho}^\kappa - \Gamma_{\beta\rho}^\sigma\Gamma_{\gamma\sigma}^\kappa) \\
& + (2 + 4c_2 + 12c_3)\delta_\alpha^\mu g_{\tau\gamma}g^{\rho\nu}(\partial_\delta\Gamma_{\beta\rho}^\tau + \Gamma_{\beta\rho}^\sigma\Gamma_{\delta\sigma}^\tau) \\
& - (8 + 20c_2 + 44c_3)\delta_\alpha^\mu g_{\tau\sigma}g^{\rho\nu}\Gamma_{\gamma\rho}^\sigma\Gamma_{\delta\beta}^\tau \\
& - (4 + 16c_2 + 32c_3)\delta_\alpha^\mu\Gamma_{\beta\delta}^\rho\Gamma_{\gamma\rho}^\nu - (2c_2 + 4c_3)g_{\beta\delta}g^{\sigma\nu}\Gamma_{\alpha\sigma}^\rho\Gamma_{\gamma\rho}^\mu \\
& + (2c_2 + 4c_3)\delta_\alpha^\mu g_{\beta\delta}g^{\rho\nu}(\partial_\rho\Gamma_{\gamma\sigma}^\sigma - \partial_\sigma\Gamma_{\gamma\rho}^\sigma - \Gamma_{\gamma\rho}^\sigma\Gamma_{\tau\sigma}^\tau + \Gamma_{\rho\kappa}^\sigma\Gamma_{\gamma\sigma}^\kappa) \\
& + \frac{6 + 28c_2 + 56c_3}{l^2}(g^{\mu\nu}g_{\alpha\gamma}g_{\beta\delta} + 4g_{\beta\delta}\delta_\gamma^\nu\delta_\alpha^\mu) . \tag{4.8}
\end{aligned}$$

The equation of motion obtained by varying the action (4.2) with respect to the spin connection is given by

$$T_{ac}{}^c e_b^\mu - T_{bc}{}^c e_a^\mu - T_{ab}{}^\mu = S_{ab}{}^\mu , \tag{4.9}$$

where

$$\begin{aligned}
S_{ab}{}^\mu &= -\frac{16\pi G_N}{e} \frac{\delta S_{NC}^{(2)}}{\delta \omega_\mu^{ab}} \\
&= -\frac{\theta^{\alpha\beta}\theta^{\gamma\delta}}{8l^4} \Big((2 - 4c_2 - 36c_3)\delta_\alpha^\mu\Gamma_{\beta\rho}^\rho e_{\gamma b}e_{\delta a} \\
&\quad - (1 + 10c_2 + 22c_3)\delta_\alpha^\mu\Gamma_{\gamma\rho}^\rho(e_{\beta a}e_{\delta b} - e_{\beta b}e_{\delta a}) \\
&\quad - (5 + 6c_2 - 8c_3)\delta_\alpha^\mu\Gamma_{\gamma\beta}^\sigma(e_{\sigma a}e_{\delta b} - e_{\sigma b}e_{\delta a}) \\
&\quad - (3 + 12c_2 + 20c_3)g_{\beta\delta}\Big(\Gamma_{\alpha\rho}^\rho(e_{\gamma b}e_a^\mu - e_{\gamma a}e_b^\mu) - \Gamma_{\alpha\rho}^\mu(e_{\gamma b}e_a^\rho - e_{\gamma a}e_b^\rho)\Big) \\
&\quad - (3 + 11c_2 + 18c_3)\Big(g_{\beta\delta}\Gamma_{\alpha\gamma}^\sigma(e_{\sigma b}e_a^\mu - e_{\sigma a}e_b^\mu) + g_{\beta\sigma}\Gamma_{\alpha\delta}^\sigma(e_{\gamma b}e_a^\mu - e_{\gamma a}e_b^\mu)\Big) \\
&\quad - (5 + 14c_2 + 24c_3)\delta_\alpha^\mu g_{\delta\beta}\Gamma_{\sigma\gamma}^\rho(e_{\rho a}e_b^\sigma - e_{\rho b}e_a^\sigma) - g_{\delta\sigma}\delta_\alpha^\mu\Gamma_{\gamma\nu}^\sigma(e_{\beta b}e_a^\nu - e_{\beta a}e_b^\nu) \\
&\quad + (c_2 - 4c_3)\delta_\alpha^\mu g_{\delta\sigma}\Gamma_{\beta\nu}^\sigma(e_{\gamma b}e_a^\nu - e_{\gamma a}e_b^\nu) \\
&\quad + (4 + 13c_2 + 24c_3)\delta_\alpha^\mu g_{\sigma\beta}\Gamma_{\delta\nu}^\sigma(e_{\gamma b}e_a^\nu - e_{\gamma a}e_b^\nu) \Big) . \tag{4.10}
\end{aligned}$$

The equations (4.8) and (4.10) have a very clear physics interpretation. The noncommutativity is a source curvature and torsion, i.e. flat space-time becomes curved as an effect of noncommutative corrections. Also, a torsion-free solution will develop a non-zero torsion in the presence of noncommutativity.

5 NC Minkowski space-time

In order to investigate consequences of noncommutativity in more details we analyze the NC deformation of Minkowski space-time. Minkowski space-time is a vacuum solution of the Einstein equations without the cosmological constant. Therefore, we first have to assume that $1 + c_2 + 2c_3 = 0$, that is that the cosmological constant is not

present in the zeroth order in the deformation parameter. Note that in our previous work [15, 16] we were not able to choose the value of the cosmological constant, since we only worked with the action S_{1NC} . Adding the other two actions S_{2NC} and S_{3NC} with arbitrary constants c_1, c_2, c_3 enables us to study a wider class of NC gravity solutions. Assuming that the solution is a small perturbation around the flat Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (5.1)$$

where $h_{\mu\nu}$ is a small correction of the second order in the deformation parameter $\theta^{\mu\nu}$, equation (4.8) reduces to

$$\frac{1}{2}(\partial_\sigma \partial^\nu h^{\sigma\mu} + \partial_\sigma \partial^\mu h^{\sigma\nu} - \partial^\mu \partial^\nu h - \square h^{\mu\nu}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha \partial_\beta h^{\alpha\beta} - \square h) = \tau^{\mu\nu}, \quad (5.2)$$

with

$$\tau^{\mu\nu} = \frac{11}{4l^6}(2\eta_{\alpha\gamma}\theta^{\alpha\mu}\theta^{\gamma\nu} + \frac{1}{2}g_{\alpha\gamma}g_{\beta\delta}g^{\mu\nu}\theta^{\alpha\beta}\theta^{\gamma\delta}). \quad (5.3)$$

The second equation (4.10) gives no contribution, that is the NC Minkowski space-time remains torsion-free in the second order of the deformation parameter. The small perturbation $h_{\mu\nu}$ we split into components h_{00} , h_{0j} and h_{ij} and we write equations separately for each component. Note that i, j, \dots are space indices, they take values 1, 2, 3 and we label $\psi = \delta_{ij}h^{ij}$. The 00, 0j and ij components of (5.2) are given by:

$$\triangle\psi - \partial_i \partial_j h^{ij} = 2\tau^{00}, \quad (5.4)$$

$$\partial_0 \partial_j h^{ij} - \partial_i \partial_j h^{j0} - \partial_0 \partial_i \psi + \triangle h^{0i} = 2\tau^{0i}, \quad (5.5)$$

$$\begin{aligned} & -\partial_0 \partial_i h^{0i} - \partial_k \partial_i h^{kj} - \partial_0 \partial_j h^{0i} - \partial_k \partial_j h^{ki} - \partial_i \partial_j h - \partial_0^2 h^{ij} + \triangle h^{ij} \\ & + \delta^{ij}(\partial_0^2 h^{00} + 2\partial_0 \partial_m h^{0m} + \partial_m \partial_n h^{mn}) - \delta_{ij} \square(h^{00} - \psi) = 2\tau^{ij}. \end{aligned} \quad (5.6)$$

In order to find a solution of these inhomogeneous equations we assume the following ansatz for the components of $h_{\mu\nu}$:

$$\begin{aligned} h^{00} &= d_3 \theta^{0\rho} \theta_\rho^0 r^2 + d_4 \theta^{0m} \theta^{0n} x^m x^n + d_5 \theta^{\alpha\beta} \theta_{\alpha\beta} r^2, \\ h^{ij} &= d_1 \theta^{i\rho} \theta_\rho^j r^2 + d_2 \theta^{im} \theta^{jn} x^m x^n + d_6 \delta^{ij} \theta^{\alpha\beta} \theta_{\alpha\beta} r^2 + d_7 \theta^{\alpha\beta} \theta_{\alpha\beta} x^i x^j \\ & \quad + d_9 (\theta^{i\rho} \theta_\rho^n x^n x^j + \theta^{j\rho} \theta_\rho^n x^n x^i) + d_8 \theta^{i\rho} \theta_\rho^l x^n x^l \delta_{ij}, \\ h^{0i} &= d_{10} \theta^{0\rho} \theta_\rho^i r^2 + d_{11} \theta^{0m} \theta^{in} x^m x^n + d_{12} \theta^{0l} \theta_l^n x^i x^l, \end{aligned} \quad (5.7)$$

where $r^2 = \sum_{i=1}^3 x^i x^i$ and d_1, \dots, d_{12} are arbitrary constants to be determined from equations (5.4-5.6). Inserting this ansatz into equations (5.4-5.6) leads to the following set of algebraic equations:

$$\begin{aligned} d_1 - d_9 + d_8 + 6d_6 - 3d_7 &= \frac{33}{8l^6}, \\ 4(d_1 - d_9 + d_8) + 3d_2 + 12d_6 - 6d_7 &= \frac{11}{4l^6}, \\ d_1 - d_9 + d_8 + 3d_2 &= -\frac{11}{2l^6}, \end{aligned}$$

$$\begin{aligned}
d_1 - d_9 + d_8 + d_4 &= -\frac{11}{2l^6}, \\
d_1 - d_9 + d_8 + 8d_6 - 4d_7 + 4d_5 - 2d_3 - d_4 &= \frac{11}{2l^6}, \\
d_1 - d_9 + d_8 + 4d_6 - 2d_7 + 2d_5 &= 0, \\
4d_{10} + 3d_{11} - 2d_{12} &= -\frac{11}{l^6}.
\end{aligned} \tag{5.8}$$

To start with, let us assume that both θ^{0i} and θ^{ij} are different from zero, $\theta^{0i} \neq 0$ and $\theta^{ij} \neq 0$. Then the solution of the previous set of equations is:

$$\begin{aligned}
d_2 &= -\frac{11}{6l^6}, d_4 = -\frac{11}{2l^6}, d_5 = -\frac{11}{8l^6}, d_3 = 0, \\
d_1 - d_9 + d_8 &= 0, \\
d_{10} &= -\frac{11}{4l^6} - \frac{3d_{11}}{4} + \frac{d_{12}}{2}, d_7 = 2d_6 - \frac{11}{8l^6}.
\end{aligned} \tag{5.9}$$

From (5.9) it follows that some constants will remain undetermined. The presence of undetermined constants suggests the existence of some residual symmetry. A detailed analysis of this residual symmetry we postpone for future work. In this paper we fix the undetermined constants in the following way: $d_1 = d_9 = d_8 = 0$, $d_{10} = -d_{12} = 0$ and $d_6 = -d_7$. Finally, the components of metric tensor follow:

$$\begin{aligned}
g_{00} &= 1 - \frac{11}{2l^6} \theta^{0m} \theta^{0n} x^m x^n - \frac{11}{8l^6} \theta^{\alpha\beta} \theta_{\alpha\beta} r^2, \\
g^{0i} &= -\frac{11}{3l^6} \theta^{0m} \theta^{in} x^m x^n, \\
g_{ij} &= -\delta_{ij} - \frac{11}{6l^6} \theta^{im} \theta^{jn} x^m x^n + \frac{11}{24l^6} \delta^{ij} \theta^{\alpha\beta} \theta_{\alpha\beta} r^2 - \frac{11}{24l^6} \theta^{\alpha\beta} \theta_{\alpha\beta} x^i x^j.
\end{aligned} \tag{5.10}$$

From the equation (4.8) it follows that the scalar curvature of the NC Minkowski space-time² is given by

$$R = -\frac{11}{l^6} \theta^{\alpha\beta} \theta^{\gamma\delta} \eta_{\alpha\gamma} \eta_{\beta\delta} = \text{const.} \tag{5.11}$$

This shows that the noncommutativity induces curvature. The sign of the scalar curvature will depend on the particular values of the parameter $\theta^{\alpha\beta}$. For example, if $\theta^{ij} = 0$ and $\theta^{0i} \neq 0$ then the scalar curvature R is positive. On the other hand, if $\theta^{ij} \neq 0$ and $\theta^{0i} = 0$ then the scalar curvature R is negative. The induced curvature is very small, being quadratic in $\theta^{\alpha\beta}$ and it will be difficult to measure it. However, qualitatively we showed that noncommutativity is a source of curvature, just like matter fields or the cosmological constant.

²Note that this result is unique and it does not depend neither on the way we choose the ansatz for solving the equation (4.8) in the case of Minkowski space-time not on the way we fix the undetermined constants.

The Reimann tensor for this solution can be calculated easily. A very interesting (and unexpected) observation follows: knowing the components of the Riemann tensor, the components of the metric tensor can be written as

$$\begin{aligned} g_{00} &= 1 - R_{0m0n}x^m x^n, \\ g_{0i} &= -\frac{2}{3}R_{0min}x^m x^n, \\ g_{ij} &= -\delta_{ij} - \frac{1}{3}R_{imjn}x^m x^n. \end{aligned} \quad (5.12)$$

This shows that the coordinates x^μ we started with, are Fermi normal coordinates. These coordinates are inertial coordinates of a local observer moving along a geodesic. The time coordinate x^0 is just the proper time of the observer moving along the geodesic. The space coordinates x^i are defined as affine parameters along the geodesics in the hypersurface orthogonal the actual geodesic of the observer. Unlike Riemann normal coordinates which can be constructed in a small neighborhood of a point, Fermi normal coordinates can be constructed in a small neighborhood of a geodesic, that is inside a small cylinder surrounding the geodesic [20]. Along the geodesic these coordinates are inertial, that is

$$g_{\mu\nu}|_{geod.} = \eta_{\mu\nu}, \quad \partial_\rho g_{\mu\nu}|_{geod.} = 0. \quad (5.13)$$

The measurements performed by the local observer moving along the geodesic are described in the Fermi normal coordinates. Especially, she/he is the one that measures $\theta^{\alpha\beta}$ to be constant! In any other reference frame (any other coordinate system) observers will measure $\theta^{\alpha\beta}$ different from constant.

6 Conclusions

In this paper we constructed a NC gravity model based on the $SO(2,3)_\star$ gauge symmetry. We used the \star -product and the enveloping algebra approach and the SW map. An effective NC gravity action was constructed using the expansion in the small NC parameter $\theta^{\alpha\beta}$. The zeroth order of the action is the commutative action (2.20). The first order correction vanishes. The second order correction is calculated; the calculation and the result are long and cumbersome. Therefore, we chose to analyze the model sector by sector. In this paper we were interested in the low energy sector, presumable describing physics at low curvatures. In that case the action is given by (4.2). The equations of motion show that, just like ordinary matter, noncommutativity plays a role of a source for curvature and/or torsion. More explicitly, in the example of NC Minkowski space time, we explicitly calculated the curvature induced by noncommutativity and showed that in the presence of noncommutativity Minkowski space-time becomes curved with a constant scalar curvature.

In addition we gain a better understanding of the diffeomorphism symmetry breaking problem in the θ -constant NC space-time. Namely, the θ -constant deformation is naturally defined for an inertial observer. Therefore, it is not possible to apply the

θ -constant deformation for GR solutions in arbitrary coordinates. With this observation we now understand the breaking of diffeomorphism symmetry in the following way: there is a preferred reference system defined by the Fermi normal coordinates and the NC parameter $\theta^{\alpha\beta}$ is constant in that particular reference system. In an arbitrary reference system the NC deformation is obtained by an appropriate coordinate transformation. We conclude that the constant NC deformation is consistent only with the reference system given by the Fermi normal coordinates. In our future work we plan to investigate other solutions of our NC gravity model, such as the NC Schwarzschild solution and cosmological solutions. Especially, we are interested in the role of Fermi normal coordinates in these solutions and in this way we hope to gain a better understanding of NC gravity. Also, using the advantage of the NC gauge theory approach we plan to include matter fields in our analysis and see the consequences of the noncommutativity on the matter part of the gravity action.

Acknowledgement We would like to thank Ilija Simonović, Milutin Blagojević, Maja Burić, Dragoljub Gočanin and Nikola Konjik for fruitful discussion and useful comments. The work is supported by project ON171031 of the Serbian Ministry of Education and Science and partially supported by the Action MP1405 QSPACE from the European Cooperation in Science and Technology (COST).

A Notation

• Metric conventions

We use the "mostly minus" or West Coast metric convention. The Minkowski metric is then

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \quad (\text{A.1})$$

The Reimann tensor is defined as:

$$R_{\mu\nu}{}^{\rho}{}_{\lambda} = R_{\mu\nu}{}^{ab} e_a^{\rho} e_{\lambda b} = \partial_{\mu} \Gamma_{\nu\lambda}^{\rho} - \partial_{\nu} \Gamma_{\mu\lambda}^{\rho} + \Gamma_{\mu\kappa}^{\rho} \Gamma_{\nu\lambda}^{\kappa} - \Gamma_{\nu\kappa}^{\rho} \Gamma_{\mu\lambda}^{\kappa}. \quad (\text{A.2})$$

Ricci tensor is defined as the following contraction of the Riemann tensor

$$R_{\nu\lambda} = R_{\mu\nu}{}^{\mu}{}_{\lambda}. \quad (\text{A.3})$$

Then the scalar curvature follows as:

$$R = g^{\mu\nu} R_{\mu\nu}. \quad (\text{A.4})$$

Note that with these conventions the scalar curvature of the vacuum solution with the positive cosmological constant (dS space) is negative, while the scalar curvature of the vacuum solution with the negative cosmological solution (AdS space) is positive.

- **AdS algebra and the γ -matrices**

Algebra relations³:

$$\begin{aligned}
\{M_{AB}, \Gamma_C\} &= i\epsilon_{ABCDE}M^{DE}, \\
\{M_{AB}, M_{CD}\} &= \frac{i}{2}\epsilon_{ABCDE}\Gamma^E + \frac{1}{2}(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}), \\
[M_{AB}, \Gamma_C] &= i(\eta_{BC}\Gamma_A - \eta_{AC}\Gamma_B), \\
\Gamma_A^\dagger &= -\gamma_0\Gamma_A\gamma_0, \\
M_{AB}^\dagger &= \gamma_0M_{AB}\gamma_0.
\end{aligned} \tag{A.5}$$

Identities with traces:

$$\begin{aligned}
\text{Tr}(\Gamma_A\Gamma_B) &= 4\eta_{AB}, \\
\text{Tr}(\Gamma_A) &= \text{Tr}(\Gamma_A\Gamma_B\Gamma_C) = 0, \\
\text{Tr}(\Gamma_A\Gamma_B\Gamma_C\Gamma_D) &= 4(\eta_{AB}\eta_{CD} - \eta_{AC}\eta_{BD} + \eta_{AD}\eta_{CB}), \\
\text{Tr}(\Gamma_A\Gamma_B\Gamma_C\Gamma_D\Gamma_E) &= -4i\epsilon_{ABCDE}, \\
\text{Tr}(M_{AB}M_{CD}\Gamma_E) &= i\epsilon_{ABCDE}, \\
\text{Tr}(M_{AB}M_{CD}) &= -\eta_{AD}\eta_{CB} + \eta_{AC}\eta_{BD}.
\end{aligned} \tag{A.6}$$

B Expanding actions: useful formulae

- **Some results used in calculations of $S_{iNC}^{(1)}$ with $i = 1, 2, 3$:**

$$\begin{aligned}
(\hat{\phi} \star \hat{F}_{\mu\nu})^{(1)} &= -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)(\phi F_{\mu\nu})\} \\
&\quad + \frac{i}{2}\theta^{\alpha\beta}D_\alpha\phi D_\beta F_{\mu\nu} + \frac{1}{2}\theta^{\alpha\beta}\phi\{F_{\mu\alpha}, F_{\nu\beta}\},
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
(\hat{D}_\rho\hat{\phi} \star \hat{D}_\sigma\hat{\phi})^{(1)} &= -\frac{1}{4}\theta^{\omega\beta}\{A_\alpha, (\partial_\beta + D_\beta)(D_\rho\phi D_\sigma\phi)\} \\
&\quad + \frac{i}{2}\theta^{\alpha\beta}(D_\alpha D_\rho\phi)(D_\beta D_\sigma\phi) + \frac{1}{2}\theta^{\alpha\beta}\{F_{\alpha\rho}, D_\beta\phi\}D_\sigma\phi \\
&\quad + \frac{1}{2}\theta^{\alpha\beta}D_\rho\phi\{F_{\alpha\sigma}, D_\beta\phi\}.
\end{aligned} \tag{B.2}$$

Under the integral, using the partial integration, terms of the form $\{A_\alpha, (\partial_\beta + D_\beta)Y\}$, where the field Y transforms in the adjoint representation $\delta_\epsilon Y = i[\epsilon, Y]$ can be covariantized:

$$\int d^4x \theta^{\alpha\beta} \text{Tr}(\{A_\alpha, (\partial_\beta + D_\beta)Y\}) = \int d^4x \theta^{\alpha\beta} \text{Tr}(\{F_{\alpha\beta}, Y\}). \tag{B.3}$$

³ $\epsilon^{01235} = +1, \epsilon^{0123} = 1$

- **Some results used in calculations of $S_{iNC}^{(2)}$:**

$$\begin{aligned}
((\hat{D}_\alpha D_\rho \hat{\phi}) \star (\hat{D}_\beta \hat{D}_\sigma \hat{\phi}))^{(1)} &= -\frac{1}{4} \theta^{\gamma\delta} \{\omega_\gamma, (\partial_\delta + D_\delta)((D_\alpha D_\rho \phi)(D_\beta D_\sigma \phi))\} \\
&\quad + \frac{i}{2} \theta^{\gamma\delta} D_\gamma (D_\alpha D_\rho \phi) D_\delta (D_\beta D_\sigma \phi) \\
&\quad + \frac{1}{2} \theta^{\gamma\delta} \{F_{\gamma\alpha}, (D_\delta D_\rho \phi)\} D_\beta D_\sigma \phi \\
&\quad + \frac{1}{2} \theta^{\gamma\delta} D_\alpha (\{F_{\gamma\rho}, D_\delta \phi\}) (D_\beta D_\sigma \phi) \\
&\quad + \frac{1}{2} \theta^{\gamma\delta} (D_\alpha D_\rho \phi) \{F_{\gamma\beta}, (D_\delta D_\sigma \phi)\} \\
&\quad + \frac{1}{2} \theta^{\gamma\delta} (D_\alpha D_\rho \phi) D_\beta (\{F_{\gamma\sigma}, D_\delta \phi\}), \tag{B.4}
\end{aligned}$$

$$\begin{aligned}
(\hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} \star \hat{\phi})^{(1)} &= -\frac{1}{4} \theta^{\alpha\beta} \{\omega_\alpha, D_\beta (D_\rho \phi D_\sigma \phi)\} \\
&\quad + \frac{i}{2} \theta^{\alpha\beta} D_\alpha (D_\rho \phi D_\sigma \phi) D_\beta \phi + \frac{1}{2} \theta^{\alpha\beta} D_\rho \phi \{F_{\alpha\sigma}, D_\beta \phi\} \phi \\
&\quad + \frac{1}{2} \theta^{\alpha\beta} \{F_{\alpha\rho}, D_\beta \phi\} + \frac{i}{2} \theta^{\alpha\beta} (D_\alpha D_\rho \phi) (D_\beta D_\nu \phi) \phi. \tag{B.5}
\end{aligned}$$

The product of (B.1) and (B.4) in the first order is given by

$$\begin{aligned}
&\left(\hat{\phi} \star \hat{F}_{\mu\nu} \star (\hat{D}_\alpha \hat{D}_\rho \hat{\phi}) \star (\hat{D}_\beta \hat{D}_\sigma \hat{\phi}) \right)^{(1)} = \left(\hat{\phi} \star \hat{F}_{\mu\nu} \right)^{(1)} (D_\alpha D_\rho \phi) (D_\beta D_\sigma \phi) \\
&\quad + \phi F_{\mu\nu} \phi \left((\hat{D}_\alpha D_\rho \hat{\phi}) \star (\hat{D}_\beta \hat{D}_\sigma \hat{\phi}) \right)^{(1)} + \frac{i}{2} \theta^{\gamma\delta} \partial_\gamma (\phi F_{\mu\nu}) \partial_\delta (D_\alpha D_\rho \phi D_\beta D_\sigma \phi) \\
&= -\frac{1}{4} \theta^{\gamma\delta} \{\omega_\gamma, (\partial_\delta + D_\delta)(\phi F_{\mu\nu} D_\alpha D_\rho \phi D_\beta D_\sigma \phi)\} \\
&\quad + \frac{i}{2} \theta^{\gamma\delta} D_\gamma (\phi F_{\mu\nu}) D_\delta (D_\alpha D_\rho \phi D_\beta D_\sigma \phi) \\
&\quad + \left(\frac{i}{2} D_\gamma \phi D_\delta F_{\mu\nu} + \frac{1}{2} \phi \{F_{\mu\gamma}, F_{\nu\delta}\} \right) (D_\alpha D_\rho \phi) (D_\beta D_\sigma \phi) \\
&\quad + \phi F_{\mu\nu} \left(\frac{i}{2} D_\gamma (D_\alpha D_\rho \phi) D_\delta (D_\beta D_\sigma \phi) + \frac{1}{2} \{F_{\gamma\alpha}, (D_\delta D_\rho \phi)\} (D_\beta D_\sigma \phi) \right. \\
&\quad \left. + \frac{1}{2} D_\alpha (\{F_{\gamma\rho}, D_\delta \phi\}) (D_\beta D_\sigma \phi) + \frac{1}{2} (D_\alpha D_\rho \phi) \{F_{\gamma\beta}, (D_\delta D_\sigma \phi)\} \right. \\
&\quad \left. + \frac{1}{2} (D_\alpha D_\rho \phi) D_\beta (\{F_{\gamma\sigma}, D_\delta \phi\}) \right). \tag{B.6}
\end{aligned}$$

The noncovariant term in (B.6) under the integral is rewritten using (B.3). Then the fully covariant form of the expanded term (B.6) is obtained:

$$\begin{aligned}
&\left(\hat{\phi} \star \hat{F}_{\mu\nu} \star (\hat{D}_\alpha \hat{D}_\rho \hat{\phi}) \star (\hat{D}_\beta \hat{D}_\sigma \hat{\phi}) \right)^{(1)} = \frac{i}{4} \theta^{\alpha\beta} \theta^{\gamma\delta} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left\{ \right. \\
&\quad \left. -\frac{1}{4} \{F_{\gamma\delta}, \phi F_{\mu\nu} (D_\alpha D_\rho \phi) (D_\beta D_\sigma \phi)\} + \frac{i}{2} D_\gamma (\phi F_{\mu\nu}) D_\delta ((D_\alpha D_\rho \phi) (D_\beta D_\sigma \phi)) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{i}{2} D_\gamma \phi D_\delta F_{\mu\nu} + \frac{1}{2} \phi \{F_{\mu\gamma}, F_{\nu\delta}\} \right) (D_\alpha D_\rho \phi) (D_\beta D_\sigma \phi) \\
& + \phi F_{\mu\nu} \left(\frac{i}{2} D_\gamma (D_\alpha D_\rho \phi) D_\delta (D_\beta D_\sigma \phi) + \frac{1}{2} \{F_{\gamma\alpha}, (D_\delta D_\rho \phi)\} (D_\beta D_\sigma \phi) + \right. \\
& + \frac{1}{2} D_\alpha (\{F_{\gamma\rho}, D_\delta \phi\}) (D_\beta D_\sigma \phi) + \frac{1}{2} (D_\alpha D_\rho \phi) \{F_{\gamma\beta}, (D_\delta D_\sigma \phi)\} \\
& \left. + \frac{1}{2} (D_\alpha D_\rho \phi) D_\beta (\{F_{\gamma\sigma}, D_\delta \phi\}) \right). \tag{B.7}
\end{aligned}$$

The remaining terms in (3.19) are calculated following the same steps.

C Variation of the action S_{NC}

Here we write some useful formulas for calculating equations of motion:

$$\delta_\omega R_{\alpha\beta}^{ab} = \nabla_\alpha \delta \omega_\beta^{ab} - \nabla_\beta \delta \omega_\alpha^{ab}, \tag{C.1}$$

$$\delta_e R_{\alpha\beta}^{ab} = 0, \tag{C.2}$$

$$\delta_\omega T_{\alpha\beta}^a = -e_{\alpha b} \delta \omega_\beta^{ab} + e_{\beta b} \delta \omega_\alpha^{ab}, \tag{C.3}$$

$$\delta_e T_{\alpha\beta}^a = \nabla_\alpha \delta e_\beta^a - \nabla_\beta \delta e_\alpha^a, \tag{C.4}$$

$$\delta_e e = e e_a^\mu \delta e_\mu^a, \quad \delta_e g_{\rho\sigma} = (\delta_\rho^\mu e_{\sigma a} + \delta_\sigma^\mu e_{\rho a}) \delta e_\mu^a. \tag{C.5}$$

References

- [1] B.P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), *Observation of Gravitational Waves from a Binary Black Hole Merger*, Phys. Rev. Lett. **116**, 061102 (2016), [arXiv:1602.03837].
B.P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), *GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence* Phys. Rev. Lett. **116**, 241103 (2016).
- [2] ATLAS Collaboration, *Search for resonances in diphoton events at $\sqrt{s} = 13\text{TeV}$ with the ATLAS detector*, arXiv:1606.03833.
CMS Collaboration, *Search for resonant production of high-mass photon pairs in proton-proton collisions at $\sqrt{s} = 8$ and 13TeV* , arXiv:1606.04093.
ATLAS and CMS Collaborations, *Combined Measurement of the Higgs Boson Mass in pp Collisions at $\sqrt{s} = 7$ and 8TeV with the ATLAS and CMS Experiments*, Phys. Rev. Lett. **114**, 33 (2015), [arXiv:1503.07589].
- [3] P. Aschieri, C. Blohmann, M. Dimitrijević, F. Meyer, P. Schupp and J. Wess, *A Gravity Theory on Noncommutative Spaces*, Class. Quant. Grav. **22**, 3511 (2005), [hep-th/0504183].
P. Aschieri, M. Dimitrijević, F. Meyer and J. Wess, *Noncommutative Geometry and Gravity*, Class. Quant. Grav. **23**, 1883 (2006), [hep-th/0510059].

- [4] T. Ohl and A. Schenckel, *Cosmological and Black Hole Spacetimes in Twisted Noncommutative Gravity*, JHEP **0910** (2009) 052, [arXiv: 0906.2730].
P. Aschieri and L. Castellani, *Noncommutative Gravity Solutions*, J. Geom. Phys. **60**, 375-393 (2010), [arXiv:0906.2774].
- [5] M. Chaichian, P.P. Kulish, K. Nishijima, A. Tureanu, *On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT*, Physics Letters B 604 1-2 (2004), [hep-th/0408069].
M. Chaichian, P. Presnajder, A. Tureanu, *New Concept of Relativistic Invariance in Noncommutative Space- Time: Twisted Poincar Symmetry and Its Implications*, Phys. Rev. Lett. 94, 151602 (2005) [hep-th/0409096].
- [6] H. S. Yang, *Emergent gravity from noncommutative spacetime*, Int. J. Mod. Phys. **A24**, 4473 (2009), [hep-th/0611174].
H. Steinacker, *Emergent Geometry and Gravity from Matrix Models: an Introduction*, Class. Quant. Grav. **27**, 133001 (2010), [arXiv:1003.4134].
- [7] M. Burić and J. Madore, *Spherically Symmetric Noncommutative Space: $d = 4$* , Eur. Phys. J. **C58**, 347 (2008), [arXiv: 0807.0960].
M. Burić and J. Madore, *On noncommutative spherically symmetric spaces*, arXiv:1401.3652.
L. Tomassini, S. Viaggiu, *Building non-commutative spacetimes at the Planck length for Friedmann flat cosmologies*, Class. Quant. Grav. **31** 185001 (2014), [arXiv:1308.2767].
- [8] A. H. Chamseeddine, *Deforming Einstein's gravity*, Phys. Lett. B **504** 33 (2001), [hep-th/0009153].
M. A. Cardella and D. Zanon, *Noncommutative deformation of four-dimensional gravity*, Class. Quant. Grav. **20**, L95 (2003), [hep-th/0212071].
- [9] P. Aschieri and L. Castellani, *Noncommutative $D = 4$ gravity coupled to fermions* JHEP, **0906**, 086 (2009), [arXiv:0902.3823].
- [10] P. Aschieri and L. Castellani, *Noncommutative supergravity in $D = 3$ and $D = 4$* , JHEP **0906**, 087 (2009), [arXiv:0902.3823].
L. Castellani, *Chern-Simons supergravities, with a twist*, JHEP **1307**, 133 (2013), [arXiv:1305.1566].
- [11] M. Dobrski, *Background independent noncommutative gravity from Fedosov quantization of endomorphism bundle*, arXiv:1512.04504.
M. Dobrski, *On some models of geometric noncommutative general relativity*, Phys. Rev. D **84**, 065005 (2011), [arXiv:1011.0165].

- [12] A. Kobakhidze, C. Lagger and A. Manning, *Constraining noncommutative space-time from GW150914*, Phys. Rev. D **94** 064033 (2016), [arXiv:1607.03776],
D. Klammer and H. Steinacker, *Cosmological solutions of emergent noncommutative gravity*, Phys. Rev. Lett. **102** (2009) 221301, [arXiv:0903.0986].
E. Harikumar and V. O. Rivelles, *Noncommutative Gravity*, Class. Quant. Grav. **23**, 7551-7560 (2006), [hep-th/0607115].
- [13] M. Burić, T. Grammatikopoulos, J. Madore, G. Zoupanos, *Gravity and the Structure of Noncommutative Algebras*, JHEP **0604** 054, 2006, [hep-th/0603044].
M. Burić, J. Madore, G. Zoupanos, *The Energy-momentum of a Poisson structure*, Eur. Phys. J. **C 55** 489-498, 2008, [arXiv:0709.3159].
- [14] M. Dimitrijević Ćirić, B. Nikolić and V. Radovanović, *Noncommutative gravity and the relevance of the θ -constant deformation*, arXiv:1609.06469.
- [15] M. Dimitrijević, V. Radovanović and H. Štefančić, *AdS-inspired noncommutative gravity on the Moyal plane*, Phys. Rev. D **86**, 105041 (2012), [arXiv:1207.4675].
- [16] M. Dimitrijević and V. Radovanović, *Noncommutative $SO(2,3)$ gauge theory and noncommutative gravity*, Phys. Rev. D **89**, 125021 (2014), [arXiv:1404.4213].
- [17] F. Wilczek, *Riemann-Einstein structure from volume and gauge symmetry*, Phys. Rev. Lett. **80** (1998) 4851-4854, [hep-th/9801184].
- [18] K. S. Stelle and P. C. West, *Spontaneously broken de Sitter symmetry and the gravitational holonomy group*, Phys. Rev D **21**, 1466 (1980).
S. W. MacDowell and F. Mansouri, *Unified geometrical theory of gravity and supergravity*, Phys. Rev. Lett. **38**, 739 (1977).
P. K. Townsend, *Small-scale structure of spacetime as the origin of the gravitation constant*, Phys. Rev. D **15**, 2795 (1977).
- [19] B. Jurčo, L. Möller, S. Schraml, P. Schupp and J. Wess, *Construction of non-Abelian gauge theories on noncommutative spaces*, Eur. Phys. J. **C21**, 383 (2001), [hep-th/0104153].
N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **09**, 032 (1999), [hep-th/9908142].
- [20] F.K. Manasse and C.W. Misner, *Fermi Normal Coordinates and Some Basic Concepts in Differential Geometry*, J. Math. Phys. **4** (1963) 735-745.
C. Chicone and B. Mashoon, *Explicit Fermi coordinates and tidal dynamics in de Sitter and Godel spacetimes*, Phys. Rev. **D 74** (2006) 064019, [gr-qc/0511129].
D. Klein and E. Randles, *Fermi coordinates, simultaneity, and expanding space in Robertson-Walker cosmologies*, Annales Henri Poincare **12** (2011) 303-328, [arXiv:1010.0588].